Introduction to Programming with Arrays using ELI

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Contents

The process of preparing programs for a digital computer is especially attractive, not only because it can be economically and scientifically rewarding, but it can be an aesthetic experience much like composing poetry or music.

-D. Knuth, the Art of Programming

By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race.

-A.N. Whitehead, quoted in Ken Iverson's Turing Lecture

Preface

The mathematician and computer scientist Jacob T. Schwartz once remarked that programming (and programming languages) consists of two sides: internal and external. The internal side is of a mathematical nature concerning the algorithmic transformation of data. The external side is of a mundane nature concerning system interface and human factors. In the programming language APL, designed by Ken Iverson, language idiosyncrasies arising from the external side have been kept to a bare minimum so a programmer can concentrate his effort on the algorithmic side of a programming job, and its programming environment is particularly simple. APL treats arrays as its primary data structure, and provides a comprehensive set of array operations as language primitives each denoted by one character in a special font (APL font). As Alan Perlis once pointed out APL encourages a *dataflow style of programming*, i.e. organizing tasks in chains of functions and operators on arrays, or what we call *array-oriented programming*. This combination of succinct notation and powerful primitives allows an APL programmer to have a clearer global view of his programming constructs than in any other language, with the possible exception of SETL, since many low level details have been suppressed.

While the APL font used to denote primitives achieves succinctness with exquisite beauty it presents difficulties for input/output, exchange of code segments in mails, and APL systems often do not communicate easily with ASCII-based text files. ELI is an array programming language based on APL where primitive operations are denoted by one or two ASCII characters thus maintaining the one-character one symbol principle of APL to preserves APL's *dataflow style in array-oriented programming*. ELI has the same programming environment centered on a *workspace* as that in APL, it also makes input/output of ASCII-based text files containing code as well as data much easier. ELI has all the language features of the classical APL [9]; it does not have the general array feature of APL2 [13] but it provides *lists* and basic operations on lists to deal with *irregular* or *non-homogeneous data*. In addition, ELI supports *complex numbers*, *symbol type* and *temporal data*.

Any discussion of program design inevitably brings up the question of *efficiency*. There are actually two different kinds of efficiency which are of concern here: the human *programming efficiency* and the machine *execution efficiency*. In other words, we like to know how quickly a working program can be implemented, and how fast that program will run. At the heart of execution efficiency is the choice of algorithm(s) used for a programming task as the design and analysis of algorithms is at the foundation of computer science [8]. However, we assume here that an efficient algorithm is already chosen which is appropriate for given circumstance in terms of likely input data size or time limit for implementation (the asymptotic efficiency of worst-case behavior of an algorithm may not be as important as the average performance of an algorithm, and one may trade a more efficient algorithm with a less efficient one in order to avoid implementation complexity). With this assumption, the execution efficiency of a program then largely depends on the efficiency provided by the language processor deployed in the process. And clearly there is a trade-off between the pursuit of execution efficiency and that of programming efficiency for each programming job. For example, one may choose to use MATLAB to solve an engineering problem instead of using FORTRAN. On the other hand, for a problem which will likely result in long running time, one will consider using FORTRAN instead of MATLAB seriously. There are two camps of programming languages, those based on

interpreters and those based on *compilers*, i.e. interpretative languages and compiled languages. In general, it is faster to develop programs in an interpretative language than in a compiled language, but a compiled program typically run much faster than its counterpart written in an interpreted language.

Iverson intended APL to be *a tool of thought* for communicating algorithmic ideas precisely (see his Turing Award Lecture [10]); consequently, APL, with mathematically inspired notations and high-level primitives, is remarkably productive in turning algorithmic ideas into programs. Arthur Whitney, developer of *A+* at Morgan Stanley (www.aplusdev.org) and *kdb* (www.kx.com), once remarked that often the *true* productivity factor provided by APL is not *5* or *10* but *infinity* because some complex systems written in APL/A+ would never have reached operational state had some conventional language been chosen to implement it. APL is also quite *versatile* as it has been used profitably in areas ranging from finance, actuarial, computer-aided design, logistics manufacturing, and research in physics, econometrics and biometrics. ELI inherits these strengths of APL. Finally, while APL programs incur the inefficiency of an APL interpreter, ELI system provides a translator (covering most array portion of ELI) to turn an ELI program into a C program to be compiled, thus avoids the execution inefficiency of its interpreter [2].

Any reader who has the patience to go through a good portion of this introduction to programming with arrays will discover that ELI is fairly easy to learn. No prior knowledge of APL or programming experience in another programming language is required. In fact, this book is quite suitable for people who have never taken a class in programming but intend to learn serious programming with only a basic background in mathematics. For people who are already familiar with some popular programming languages such as C, Java, we point out that in C/C++ or Java a program is mostly organized around loops; in contrast, a program in APL/ELI is best organized as chains of powerful primitives manipulating arrays. This suppression of details not only results in elegant APL/ELI programs but also eliminates a lot of low level cleric errors in programming. Moreover, such a source program is ideal for automatic parallelization [6] by a parallelizing compiler on multi-core machines. Finally, one must have programmed in APL/ELI to fully appreciate the ingenuity of Iverson in designing APL: its economy of notations, innovative treatment of limiting cases and remarkable mathematical consistency. As ELI is freely available on multiple platforms, we hope this introduction will attract more people to experiment with ELI and come to realize that programming with arrays can indeed be an aesthetic experience much like composing poetry or music, and that a clean notation indeed sets our brain free to concentrate on more advanced problems.

Mount Kisco, NY, 2014.

1. Array, List and Primitive Operations in ELI

1.1 Computers and Programming Languages

A *computing device* is some object which can carry out computations reliably. Today such devices range from smartphones, tablets, personal computers, servers to supercomputers. For simplicity, we call all these computing devices *computers*. A basic computer consists of three units: *central processing unit* (CPU), *memory* and *input/ output* unit (or simply I/O); this is the classical *von Neumann machine*. Today, a parallel computer's processing unit can starts from multi-core processor to multi-processors, and there are shared-memory parallel computers as well as distributed-memory parallel computers with various inter-connection networks. An input device can be a touch screen or a keyboard; an output device can be a liquid crystal display or a laser printer. The memory unit is where a computer stores users' programs and data. The CPU fetches an instruction-stream from the memory, decodes it, i.e. figures out the meaning of the instructions, which may include fetching data from memory, and carries out its execution in an appropriate order. Each computer understands only a fixed set of instructions in binary form, i.e. some particular sequences of 0's and 1's. For example, an Intel-based PC understands the Intel x86 Instruction Set. This set of instructions constitutes the machine language of an Intel-based computer.

It is certainly extremely tedious and error-prone to prepare a program in the machine language of a computer. So the first thing people did was to introduce assembly languages. An assembly language for a particular type of computer is basically the same as the machine language of that class of computers except for the following two points: First, *mnemonics* are used to represent operation code instead of their binary format. For example,

```
A
```

means the *add integer* instruction instead of the binary string

```
01011010
```

which the machine (an IBM 370) understands. Second, *labels* are introduced to denote a certain place for an instruction, and names are introduced to represent numbers which are memory locations. For example, if

```
L1: A 1,2
```

is the 99-th instruction in a program, an instruction

```
B L1
```

will branch to that instruction without specifying that the instruction with label L1 is the 99-th instruction.

The *memory* of a computer can be visualized as a contiguous array of numbered boxes (i.e. each box has an address), each containing a binary data of 8 bits (a *bit* is the basic electronic entity and can be either *on* or *off* to represent a binary digit of `1` or `0`). Such a box is by custom called a *byte*. Memory in a computer usually is only addressable by *words*; a word can be either 4 bytes or 8 bytes. A byte can have 256 different configurations and can therefore be used, via some encoding scheme, to present a set of 256 characters. For example, in a commonly used encoding system called ASCII the character '`A`' is represented by

```
01000001
```

Usually, an integer is represented in its binary form in a word with the first bit indicating the sign. For example, 3 and -2 are represented in 32-bit word by

```
00000000000000000000000000000011
```

```
11111111111111111111111111111110
```

respectively. Numbers with possible fractional parts are represented differently in floating-point format which we will not get into here. Besides characters and numbers, a byte or a word of data can represent other entities through various representation schemes.

The CPU of a typical computer consists of a *control unit*, an *instruction decoder*, an *address generator*, a *program counter*, *arithmetic and logic unit* (ALU), a *program status word* (PSW) which is used to record the *state* of the machine and is saved when machine execution is interrupted, part of PSW is called *condition code*, and several *general purpose registers* denoted as

```
R0, R1, …, R15
```

The registers are simply data storage places (16 in IBM 370 with 32 bits each) located in CPU. The execution of a program, i.e. a sequence of machine instructions is as follows:

1. Fetch the next instruction to CPU and increase the program counter so it points to the next instruction (adds 4 if each instruction is 4 bytes long).
2. Decode the instruction.
3. If the instruction involves data from memory then fetch the data at the memory location pointed to by the address generator.
4. Execute the instruction and go back to step 1.

Soon after assembly languages were created, people discovered that it was helpful to introduce a convention so that a short-hand notation like an assembly instruction can replace a repeatedly used sequence of code, with possible variation of data. This is an *assembler with macro-facility*. Life is made a bit easier, but it is still quite a burden if you always have to keep track of registers and memory locations during programming.

Finally, people introduced *high-level programming languages*. Instead of operating on registers and memory locations, *variables* are used for data items. Consequently, program statements can be written more like ordinary mathematical computations. But machines do not understand any of the high-level languages. So people write programs to bridge the gap between the machine (or assembly) languages and various high level languages. Such programs are called *language processors*. Among high level programming languages, there are, like computers, *general purpose* programming languages designed for a variety of applications, and *special purpose* languages designed for some specific application areas. We shall only talk about general purpose programming languages.

There are two kinds of programming language processors: compilers and interpreters. A ***compiler*** COMP for a programming language L for a machine M accepts a program *p*[L] written in programming language L as an input to COMP, and transforms *p*[L] into a semantically equivalent program *p*[Ml] written in the assembly language Ml of computer M as the output of COMP.

COMP: *p*[L] –> *p*[Ml]

The program (text) *p*[L] is usually called the *source* code while *p*[Ml] is called the *compiled* code which can further be turned into object code *o*[Ml] of machine M by an assembler for Ml. When we run *o*[Ml] on machine M, it would accept what *p*[L] would accept as input and carry out operations specified in *p*[L] including production of outputs.

An ***interpreter*** of a programming language L for a machine M, on the other hand, is a program INTP which accepts a program *p*[L] written in L, one unit (usually an *expression* in that language) at a time, and carries out immediately what was semantically specified in that unit until the whole program *p*[L] has been processed. Languages whose processors are primarily compilers are called *compiled languages*; languages whose processors

are interpreters are called *interpreted languages*. The division becomes less clear when some compiled languages offer interpreters and some interpreted languages acquire compilers. A compiled language typically requires a programmer to *declare* in the beginning of his program all *variables* (and their *types*) to be used in his program while interpreted languages usually do not have such a requirement.

Examples of important compiled programming languages are:

- FORTRAN, the first high-level programming language, and still used in scientific community today.
- COBOL, most used programming language for commercial data processing for a long while.
- PASCAL, invented for structured programming and enforces strict typing.
- C/C++, C is the most commonly used system programming language and serves as a common assembly language; C++ is based on C but incorporating object-oriented programming features.
- ADA, a language based on PASCAL and promoted by US Department of Defense.
- Java, an object-oriented language designed to run on a wide range of computing/communication devices.

Examples of interpreted programming languages are, with each of its descendants in parentheses:

- BASIC (Visual Basic), a simplified version of FORTRAN, designed for easy learning and widely available.
- Lisp (Common Lisp, Scheme), a list based programming language used widely in artificial intelligence area.
- APL (APL2, J, Q, ELI), an array oriented programming language with succinct symbols.
- MATLAB (Scala), it is functionally similar to APL, but with FORTRAN like syntax.
- Perl, started as a language for text processing and become very popular for shell language like system work.
- Python, it is similar to Perl but with good vector processing capability.
- Ruby (Ruby on Rails), it is a programming language popular for writing web applications.
- Haskell, a functional programming language with static typing.

Strictly speaking, Java is interpreted, i.e. a Java program is compiled into a sequence of Java Virtual Machine (JVM) code, and that sequence of JVM code is then interpreted. As JVM is implemented on different machines and communication devices differently, the compiled JVM code for a particular Java program remains the same. In fact, a Perl program is also not interpreted line by line but compiled into an internal form which is then interpreted. The reason we listed Java in the compiled camp is because its programs required variable declarations similar to that in C. In the case of Perl, variable declaration is not exactly required. On the other hand, there exists working compiler for (classical) APL ([3],[5]), though not commercially available; and we do have a compiler which covers a large portion of the ELI programming language available [2]. Thus, ELI would provide both the programming convenience and productivity of an interpreted language as well as the execution efficiency of a compiled language for large portion of its programs. MATLAB also offers a compiler to produce code which can be executed independent of its interpreter environment if not necessary to provide higher execution efficiency.

We remark that many compiled languages nowadays provide Interactive Development Environment (IDE) to make their coding/debugging process more like that of an interpretive language. The most well-known IDE may be Microsoft's Visual C++, now part of Visual Studio covering other languages such as Basic. Still, interpretive languages, by saving the compilation step and skipping elaborative declaration in a program, provide appreciable programming productivity over compiled languages while suffer in execution efficiency. For many non-product application software, this is understandably a good trade-off; and further efforts in providing compilation capability for some interpretive languages just make this trade-off more attractive.

There are two ways to envision about how to structure a (large) program: the top down approach and the bottom up approach. The top down approach is best exemplified by the object-oriented programming model while the bottom up approach is best exemplified by very high level programming languages, i.e. incorporating frequently used operations on main data structures such as arrays into very high level language primitives. C++ and Java are in

the first camp while MATLAB and APL/ELI are in the second camp. There are also attempts to combine the two as shown by Scala, an open source descendent of MATLAB.

The design philosophy of Ken Iverson on APL is that of simplicity. APL is powerful in the sense expressed by Conrad Barski in ***Land of Lisp***: "*To make a programming language powerful, you need to make it expressive. Having an expressive language means that you can do a lot of stuff with very little actual code*". This approach applies not only to the economy of notation in language syntax but also in having only a few organizing concepts for the language. This is in sharp contrast with that of object-oriented programming model where one has to learn classes, methods, inheritance etc. before any meaningful example can be discussed. Admittedly, complex (system) applications need object-oriented programming model to develop and maintain manageable code, but many programming jobs do not really need such elaborate conceptual prerequisites to accomplish a job at hands. Moreover, APL and its descendants have been used to implementing huge systems in chip design, manufacturing logistics, financial databases with time series and derivative trading systems. ELI adopts all minimalist design principles of APL. ELI has made only one pragmatic compromise with respect to APL, i.e. it uses ASCII characters in lieu of the special APL character set in order to have easier exchange of code and data through e-mail or file input/output. Still, by replacing each APL character with one or two ASCII characters, ELI has essentially maintained the *one-character one-symbol* spirit of APL language. On the other hand, ELI has added list, dictionary and table to ISO APL [9] to enhance its power in handling non-homogeneous data and make it more convenient for doing large data analysis.

1.2 ELI System and its Data Types

Once you downloaded ELI and installed it on your computer, if it is a Windows system, just put *eli.exe* in your *desktop* or some other directory. Click on the *eli.exe* icon and you see a window pop up with the following lines:

```
ELI version 0.2 (C) Rapidsoft

        CLEAR WS
```

For Linux or Mac OS platform, after you put *elix* (*elim* for Mac OS) in a directory of your choice, cd to that directory and type *./elix*, you see the same two lines response as the above except that there is no GUI window as ELI is command line based for Linux and Mac OS.

Now you are in ELI, i.e. you entered the ***ELI programming environment***. To get out this ELI environment after you finish doing programming or experimentation, simply type

```
        )off
```

then you would be back to the host system, be it Windows, Linux or Mac OS. ELI, as in APL, provides *workspaces* as a basic organizing unit for programmers to develop, debug, save and load his (saved) programs. A ***workspace*** consists of *stored data* (***variables***) and (*user defined*) ***functions*** as well as errors encountered in running *expressions* (portions of a program). `CLEAR WS` indicates it is a *clear workspace* meaning that it is a clean slate to work on, i.e. there is nothing there, except some pre-existing ***system variables*** provided by the ELI system. One of such system variable is `[]IO`, the *index origin*, which can be either `1` or `0`. Type

```
        []IO
1
```

You *type* again (the lines displayed with an indentation), and you see the system *responses* with a line:

```
        !10
1 2 3 4 5 6 7 8 9 10
```

where **!** is the ***interval*** *function* which generates a vector of `n` integers from `1` or `0` depends on `[]IO`. We can change the value of `[]IO` by an assignment and see its effect on !:

```
       []IO<-0
       !10
0 1 2 3 4 5 6 7 8 9
```

Hence, for `[]IO= 0`, it is just like in C. Later we'll see that there are other primitive operations such as indexing which also depend on `[]IO`. Continue to explore, we type in

```
       100+!10
101 102 103 104 105 106 107 108 109 110
       v<-100+!10
       v
101 102 103 104 105 106 107 108 109 110
       w<-2*v
       w
202 204 206 208 210 212 214 216 218 220
```

We notice two things: i) ELI code operates from right to left and ii) value created can be stored into a ***variable*** by an ***assignment*** (`<-`). Now, if we type

```
      a
value error    (* system response *)
      a
      ^
```

This is because a variable named `a` has not been assigned any value yet. A *variable* ***name*** must start with an alphabet, then possibly followed by alphanumerical characters and the '_' character (but '_' cannot be the ending character). For example, `ab12_99`, `Acx`, `h_103` are legitimate variable names while `0ac` and `u55_` are not. Once a variable is created by an assignment, its value can be further used in later operations:

```
     2+a<-1
3
```

We note the ***assign*** *symbol* `<-` is a *two character* symbol while =, which represents the symbol for assignment in some programming languages, is the symbol for the *equality* function in ELI. Symbols which represent primitive functions, i.e. operations provided by the ELI system, consist of either one or two ASCII characters. For a two character symbol such as assignment no blank character is allowed in between. Let us continue our exploration:

```
      b<-'A'
      b
A
      b+1
domain error
      b+1
       ^
```

This is because variable `b` has a value which is of type character and arithmetic operations are not defined on data of character type. Here, by ***data*** we refer to either *literal data*, i.e. data directed represented the item as written, or *variables or expressions* holding values of various types.

ELI has four basic types of data: *numbers, character, symbol* and *temporal data*. We have just seen numeric and character data. Symbolic data are like (variable) names affixed with a back tick such as

```
   `a1 `aapl `goog
```

The system variable `[]TS` gives the *current time* which is a temporal data of type ***datetime***:

```
     []TS
2013.07.14T00:04:54.332
```

while `2013.07.14` is of type ***date*** and `00:04:54` is of type ***second***. Altogether there are six subtypes of *temporal data type* (see [4] sect. 2.4).

There are four subtypes of numerical data: *boolean*, *integer*, *floating point* and *complex numbers*, in an expanding order; boolean data consists of only `0` and `1` which represent ***false*** and ***true*** respectively. A *boolean* is also an integer, an integer can be used where a floating point number can be used and a floating point number can be used where a complex number can be used. A negative number is affixed with the '_' character:

```
     2-1 2 3
1 0 _1
```

To write a negative number the '_' character must immediately be followed by a digit; it is treated similar to the '.' point in a number. Numbers are written in decimal form

```
    24 _1.2 0.2 3.0 .5
```

are all legal representation of numbers in ELI except the last one since there must be a `0` before '.' for a fractional number less than 1 in ELI. For scaling, both '`e`' and '`E`' are acceptable:

```
       1.2e2
120
       1.2E_2
0.012
```

A ***complex number*** is of the form `RjI`, where `R` is the *real* part and `I` is the *imaginary* part, each written as an integer or a floating point number and there should be no space before or after `j`. For example the square root of `_1` is

```
  _1*.0.5
0j1
     2j5+3j2.5
5j7.5
      2j5*3j2.5
_6.5j20
```

We note that ELI does not make an explicit distinction between floating point numbers and fixed point numbers. In other word, in ELI one does not concern about how a number is represented in the machine. An *integer* is an integral number which may be represented in floating point format in the machine by the system if it is too large.

As we have seen already literal character data is a string of characters quoted by a pair of quotation marks as in the assignment above to variable `b`. But when ELI displays the value of `b` the quotation marks are taken away. To represent a quotation mark, we need to use a double quotation:

```
    ch<-'abcd''e''1234'
    ch
abcd'e'1234
```

The functions provided by the ELI system which operate on ELI data are called ***primitive functions***. We have already seen a few primitive functions: `!` (*interval generator)*, `+` (*addition*), `*` (*multiplication*) and `*`. (*power or exponent*). A primitive function is called ***monadic*** if it takes one (*right*) argument or ***dyadic*** if it takes two

arguments, i.e. a *left* and a *right* argument. Some primitive functions apply to data of any type, and some primitive functions apply only to data of certain type. For example, `!` applies only to non-negative integers while `=` applies to a pair of any data.

```
      'A'=b
1
      1=b
0
```

When a primitive function applies to *improper* data, i.e. where it is not defined, it results in a `domain error`.

ELI has two modes of operation: ***execution mode*** and ***function definition mode***. What we have seen is the *execution mode*: you type in some ELI expression, the system responses either with an answer or an error message. This is why systems like ELI are called interactive, or interpretive. The system is in *function definition mode* when you start to create and edit a user defined function (also called procedures in languages like C). We'll come to that in the next chapter. One can also prepare function text and other expressions in a file outside of the ELI system using an ordinary text editor and then load that file into the ELI system (see chapter 2).

ELI system recognizes two broad classes of instructions: ordinary ELI ***expressions*** dealing with the algorithmic calculations/transformations of data in an active workspace, and ***system commands***. We can look at an ELI system as consisting of two components: a ***supervisor*** and an ***interpreter***. The *supervisor* is the broker between the ELI programmer and the outer environment, i.e. the operating system of a machine (Windows, Linux or Mac OS) where the ELI system is situated. It takes care of the initiation and termination of an ELI work session (the)off command), saving and loading saved copies of workspaces; loading files and transferring data and functions to files. The *interpreter* parses and executes an ELI expression, and its result for which we have just seen few examples. One of the system commands is to change, or give a name to the clear workspace we have been working on:

```
)wsid abc
```

by that we give the name `abc` to the active workspace, the other is

```
)save
```

Note that a clear workspace cannot be saved; it must have a name. A saved workspace can be loaded later by

```
)load abc
```

The system command

```
)vars
```

will list all user defined variables in the current workspace, and the command

```
)vars a
```

will list all user defined variables whose name starting with `a` in the current workspace; the system command

```
)fns
```

will list all user defined functions in the current workspace, and the command

```
)fns a
```

will list all user defined functions whose name starting with `a` in the current workspace. There are other system commands (see [4]).

1.3 Shape of Data, Reshape and Data Conversion

A single data item such as a number, a character of a symbol and a variable which is assigned such a value is called a ***scalar*** while a group of items of the same type, or a variable holding such a value, is called an ***array***. For example, `1` is a scalar and the variable `b` in the previous section is holding a scalar value while the variable `ch` and (the result of) `!9` are one dimensional arrays. To know the *shape* of (literal) data or a variable, we apply the monadic primitive function `#`, called ***shape of***, to the data or variable in question. The shape of a scalar (say variable `b` above) is an *empty vector*

```
 #b
```
(* *system response is a blank indicating an empty vector* *)

i.e. scalars have no shape just like points have no length. There are two ways to write a literal empty vector: `!0` or `''`. What is the shape of an empty vector?

```
 ##b (* we ask the shape of #b *)
0
 #!0
0
 ''=!0
```
(* *system response is a blank indicating an empty vector* *)

Hence, the system says the shape of an empty vector is a vector of length `0`. Note that the response to our question of whether `''` equals to `!0` is neither *true* (`1`) nor *false* (`0`) but an empty vector. This may seem to be a surprising choice until we understand that the primitive function '`=`' requires both arguments to have the same shape `S` (or one side is a scalar, as we will explain in more detail later) and its result is always of the same shape as that of `S`.

A one dimensional array is called a *vector*, a two dimensional array is called a *matrix* and there are arrays of higher dimensions. All elements in an array must be of the same type, i.e. they are either numeric, character, symbolic or temporal. For a vector `v`, the *shape* of `v` is its length.

```
 #a<-1 3 7 9
4
 #C<-'abcde'
5
```

We see from the above that to denote a numeric vector we only need one or more spaces to separate the numbers which are the individual elements of the vector; and to denote a character vector we write it as a quoted string with no space in between unless we want to include blank characters. If we have

```
  a
1 2 3
4 5 6
```

then

```
   #a
2 3          (* system response *)
```

i.e. `a` is a matrix of 2 rows and 3 columns. If `ec` is the following 3-dimensional array

```
     ec
abcd
abcd
abcd

abcd
abcd
abcd
```

then

```
    #ec
2 3 4
```

In general, for an array `a`, `#a` is a vector `sv` whose elements are the lengths of `a` in each dimension, the *axis*. For a matrix, the first dimension runs from top to bottom and the second axis runs from left to right. `##a,` i.e. the *shape* of the *shape vector* of `a`, is called the ***rank*** of `a`, which is the *dimension* of `a`.

There is a monadic primitive function ***count*** `^`, for which `^w` gives the number of elements in `w`, and for a scalar `w`, `^w` is `1` (this saves us the need to ravel a scalar in cases for counting purpose). So we have,

```
       ^ec
24
       ^`abc
1
       ^'abcd'
4
```

A convenient way to produce some numeric vector in ELI is to use the *monadic* primitive function `!` called *interval generator*:

```
    !10
1 2 3 4 5 6 7 8 9 10
```

It gives a vector of 10 integers starting from `1` if `[]IO` is `1` (or starting from `0` and ending in `9` if `[]IO` is `0`). Another way to generate vectors is to use the dyadic primitive function `#` called ***reshape***

```
     10#'a'
aaaaaaaaaa
```

A primitive function symbol is called *ambivalent* if it denotes either a *monadic* or *dyadic* function, i.e. having one or two arguments. In ELI, for *economy* of symbols and *conciseness* of notation, a one-character (or two-character) symbol usually represents two primitive functions: one monadic and one dyadic, depending on whether there is only one argument to the right of that symbol or there are two arguments, i.e. with an additional argument to the left of the function symbol. So, `#a` is the shape of `a` while `s#a` is the reshape of `a` into an array `s#a` of shape `s`, where `s` is a non-negative integer or a vector of non-negative integers. For example,

```
      a<-3 4#!10
      a
```

```
1  2  3  4
5  6  7  8
9 10  1  2
      #a
3 4
      2 3 4#'abcd'
abcd
abcd
abcd

abcd
abcd
abcd
      0#a
      (* system response is a blank line indicating an empty vector *)
```

i.e. an empty vector, because 0 is the shape of an empty vector;

```
      3#'abcd'
abc
```

We notice that the reshape function `#` use the elements of its right operand to form an vector/array whose shape is specified by the left operand. In case there are not enough elements to go around, it would reuse previously appeared elements (hence a convenient way to generate many copies of a data item is to reshape it); and in case there are more elements than needed, it only takes the amount it needs. In summary, we have

`#s#a` ←→ `s`

namely, the *shape* of the result of a *reshape* is the *left operand* of the reshape. In ELI there are many such identities which help one to reason about a program. Let `sx<-10` and `vx<-1#sx` then `#sx` is an empty vector while `#vx` is `1`, but both `^sx` and `^vs` are `1`. Hence, in some situations the ***count*** function `^` is more convenient to use.

There is a monadic primitive function ***format*** `+.a` which will turn its numeric or symbolic operand `a` into its character representation:

```
      +.a<-12.3
12.3
      ^a
1
      ^+.a
4
      +.as<-`abc `b1
abc b1
      #as
2
      #+.as
6
```

Conversely, if we have a character string `s` which represents a number or a symbol then the ***execute*** function `!.s` will turn it into a number or symbol:

```
      2+!.'12.3'
14.3
      5#!.'`abc `b1'
`abc `b1 `abc `b1 `abc
```

The function `!.` is actually far more powerful; it takes in a character string and executes it as a line of ELI code.

1.4 Mathematical Computations

ELI has a rich set of primitive functions, i.e. more than the usual four arithmetic functions found in most programming languages, with additional ones coming from something like a standard math library. There are two kinds of primitive functions in ELI: ***scalar*** and ***mixed***. A *scalar* function has the characteristic that when `f` is applied to an array, it is an extension of `f`'s application to each element of the array for monadic `f`. For dyadic `f`, its two arguments must be *conformable*, i.e. either both are of the same shape, or one of them is a scalar or a one element vector, in which case that argument is reshaped to the shape of the other argument before function application. Suppose we have

```
      b
1 2 3 4
      c
5 6 7 8
      b+c
6 8 10 12
      100+b
101 102 103 104
      a
  1 2 3
  4 5 6
      d
12 11 10
9  8  7
    a+d
13 13 13
13 13 13
   c
1 1
1 1
1 1
  a+c
 length error    (* because a and c do not have the same shape *)
     a+c
      ^
```

Two more examples:

```
  -1 2 3 4
_1 _2 _3 _4
  5-1 2 3 4
4 3 2 1
```

Note that the monadic *negation* function applies to every element of the right argument and produces a vector of 4 negative numbers.

There are three groups of scalar functions: *arithmetic*, *logical* and *relational*. The arithmetic functions are

monadic	***symbol***	***dyadic***
conjugate	+	*add*
negative	−	*subtract*
signum	*	*multiply*
exponential	*.	*power*
reciprocal	%	*divide*
natural logarithm	%.	*general logarithm*
pi times	@	*circle functions*

absolute value	`|`	*residue*
floor	`_.`	*minimum*
ceiling	`~.`	*maximum*
roll	`?.`	*deal (a mixed function)*
factorial	`|.`	*binomial(mixed function)*

The ***domain*** of a primitive function `f` is a data (*sub-*)*type* where `f` is well-defined. For example, shape(`#`) is defined for all data types while monadic `!` is defined only for non-negative integers. The domain of arithmetic functions is numeric data in general, but further restrictions on some functions are required. A numeric function always yields a numeric result. We refer to [4] for a detailed description of these functions. We shall give some examples of the use of these functions besides the ones we have already encountered.

Let `a` and `b` be literal numeric data or variables having numeric value in the following examples. `a*b` is the product of `a` and `b`. while `a*.b` is `a` to the power of `b`.

```
   2*0 1 2 3 4 5 6
0 2 4 6 8 10 12
   2*.0 1 2 3 4 5 6
1 2 4 8 16 32 64
```

We have already seen that `-a` is `0-a` while `%a` is `1%a`.for a numeric `a`, scalar or array. We note that `-` is the inverse function of `+` while `%` is the inverse function of `*` and `0` is the identity element of + (i.e. it is neutral to the operation: `0+a=a` for all `a`) while `1` is the identity element of `*` (i.e. `1*a=a` for all `a`). `2*.0.5` is `1.414213562`, the square root of `2`, and we can compute cube root, 4th root etc. similarly:

```
      %1 2 3 4 5 6
1 0.5 0.3333333333 0.25 0.2 0.1666666667
      64*.%1 2 3 4 5 6
64 8 4 2.828427125 2.29739671 2
```

We state here the most important rule in evaluating an ELI *expression*: *evaluation is* ***from right to left*** in the sense that the result of an operation on the right feeds as an input to the next operation, and all functions have ***equal precedence*** but in case the left operand of a dyadic function is in parentheses then what inside the parentheses must be evaluated first. Hence,

```
    2*3+10
26
    (2*3)+10
16
```

Back to computing various roots of a number, we calculate reciprocals of a sequence of numbers first and then apply the power function. This is actually a special example of taking fractional exponents (powers). Next

```
    _1*.0.5
0j1
```

as we know that square root of `_1` is the complex number ***i***. A *complex number* is written in the form `RjI` where `R` is the *real part* and `I` is the *imaginary part*. Note that no blanks are allowed before or after `j` and what follows `j` must be part of a literal number, but if `I` is `0` then it reverts back to real number form. For two complex numbers, $R_1jI_1+R_2jI_2$ equals to R_3jI_3 where $R_3=R_1+R_2$, $I_3=I_1+I_2$ and $R_1jI_1*R_2jI_2$ equals to R_3jI_3 where $R_3=(R_1*R_2)-I_1*I_2$, and $I_3=R_2I_1+R_1I_2$

```
     2j5+3j2.5
5j7.5
      2j5*3j2.5
_6.5j20
```

The monadic `+` function ***conjugate*** when applies to a complex number `RjI` results in $\texttt{RjI}_{\texttt{n}}$ withere $\texttt{I}_{\texttt{n}}=-\texttt{I}$:

```
    +_1*.0.5
1
    +_1j0.5
_1j_0.5
```

We'll introduce the derived function *sum* of a vector `v` here as `+/v`:

```
      +/!100
5050
      +/v<-_1 2 3 6.8 10
20.8
```

For a vector (*x;y;z*) in 3-dimensional space, the *length* of this vector is *square roo*t of $(x^2+y^2+z^2)$. Hence if a vector `v` in ELI of n elements is representing a vector in n-dimensional space, then its *length* is calculated as

```
    (+/v*.2)*.0.5
12.65859392
```

for the `v` above. However, for a vector `u` in a complex n-dimensional space the length is computed a square root of the sum of $\texttt{u*(u}^{\texttt{*}}\texttt{)}$, where $\texttt{u}^{\texttt{*}}$ is the conjugate of `u`. For example,

```
      u<-0j1 1j2 4
      (+/u*+u)*.0.5
4.69041576
```

The monadic `*` function ***signum*** when applies to a real number `R` results in

`1` if `I>0`; `0` if `I=0`; `_1` if `I<0`

Suppose `P` is the prices of a stock trades in the first minute of market opening, `P0` (=`20`) is the closing price of that stock in the previous day. Now we can see easily which are up-trades and which are down-trades:

```
      P
17.3 22.3 24.3 17.5 21.4 18.4 17.2 15 20.8 21.8
     *P-P0<-20
_1 1 1 _1 1 _1 _1 _1 1 1
```

and this result can be used further for other purpose as we'll see later. We remark that the *signum* function is also defined for complex numbers but we are not going to explain here this mathematical extension.

The *monadic* function `*.` is the mathematical *exponential* function.

```
       eu<-*.1
       eu
2.7182818
       *. _1 0 1 2 3
0.36787944 1 2.7182818 7.3890561 20.085537
       eu*. _1 0 1 2 3
```

```
0.36787944 1 2.7182818 7.3890561 20.085537
```

We see that `eu` above is the famous mathematical constant (a transcendental number) ***e***, also called the *Euler number*, and that `*.v` is just a convenient way to write `eu*.v` for any numerical (scalar or array) `v`.

The *dyadic* function `a%.b` is the `a` based *logarithm* of `b` , and the *monadic* function `%.b` is `eu%.b`, i.e. the ***e*** based natural logarithm. For example

```
      10%.1 10 100 1000 1000 10000 100000
0 1 2 3 3 4 5
      eu*.0 1 2 3 3 4 5
1 2.718281828 7.389056099 20.08553692 20.08553692 54.59815003 148.4131591
      %.1 2.718281828 7.389056099 20.08553692 20.08553692 54.59815003 148.4131591
0 1 2 3 3 4 5
```

Note that `%.*.b` ←→ `b` ←→ `*.%.b` for a scalar or array `b`.

The *monadic* function `@a` is just ***pi*** *times* `a`:

```
      @!6
3.141592654 6.283185307 9.424777961 12.56637061 15.70796327 18.84955592
```

For a sphere of diameter r, the volume v is (4/3)*pi*r^3; let r=2, then the volume of the sphere is

```
     (4%3)*(@1)*(r<-2)*.3
33.51032164
```

The dyadic function `b@a` is actually an *encoding* of several mathematical functions (called ***circle*** *functions*), the value `b` must be an integer ranging from `_12` to `11` which determines a specific function. For example, `0@a` is the function `(1-a*.2)*.0.5`. Most circle functions are trigonometric functions and we list some common ones here (see [4] for a complete list).

ELI notation	mathematical function
`1@a`	sin `a`
`2@a`	cos `a`
`3@a`	tan `a`
`_1@a`	arcsin `a`
`_2@a`	arcos `a`
`_3@a`	arctan `a`

Now we can write the ***Euler*** *formula* in ELI expression as

```
      *.0j1*x ←→ (2@x)+0j1*1@x

      x<-@0.5 1 2
      *.0j1*x
0j1 _1 1
      (2@x)+0j1*1@x
0j1 _1 1
```

(the middle item above represents $e^{i\pi} = -1$). The monadic function `|a` is the ***absolute value*** function:

```
      |0.5 _1.2 7 9.3
0.5 1.2 7 9.3
```

The dyadic function `b|a` is the ***residue*** function: when `b=0`, `b|a` is `|a` ; otherwise `b|a` is the remainder of `a` divided by `b`:

```
     2 3 4 5|7 5 3 10
1 2 3 0
```

The monadic function `|.a` is the ***factorial*** function: for a non-negative integer `a`, `|.a` is the product of `1..a`:

```
      |.3 5 8
6 120 40320
```

We note that the dyadic function `b|.a` is not a scalar function, it is the ***binomial*** function of number of ways to take `b` items out of a set of `a` distinct items for `b<=a`, i.e.

```
b|.a ←→ (|.a)%(|.b)*|.a-b
```

1.5 Comparisons, Selection, Membership and Index of

Any two scalars or arrays of the same shape, or one scalar and one array can be compared for *equality* or *inequality* by the primitive function `=` or `~=`:

```
      'A'=2 3 4
0 0 0
      'AbC'='abc'
0 1 0
      2 3 4~=2 3.1 5.2
0 1 1
      'abc'=`abc
0 0 0
```

Clearly, if two items are of different data types such as one numeric and one character, or one character and one symbol, the result is ***false*** which is denoted by `0`, and if the result is ***true*** it is denoted by `1`. For two conforming numerical data (excluding complex numbers) or two character data they can further be compared by the following functions: *less than* (`<`), *less than or equal*(`<=`), *greater than* (`>`), *greater than or equal* (`>=`). Here for character data the comparison is determined by their *lexicographic* order.

```
     'abc'<'acb'
0 1 0
     'abc'<='acb'
1 1 0
```

All these comparison functions produce boolean results, and there are several *logical* functions operating on boolean data: *not* (monadic `~`), *and* (dyadic `^`), *or* (dyadic `&`):

```
      ~b<-0 1 0 0 1 1 1 0
1 0 1 1 0 0 0 1
      b^0 1 1 1 0 0 1 1
0 1 0 0 0 0 1 0
      b&0 1 1 1 0 0 1 1
0 1 1 1 1 1 1 1
```

All comparison and logical functions are scalar functions. The benefit of denoting *true/false* by boolean bits is we can do comparisons and logical operations one after another to succinctly produce what we want. We introduce a mixed function ***compression*** here: `x/y`, the left operand `x` must be boolean and for a right operand `y` which is a vector it *select* those elements in `y` which correspond to a `1` in `x`. For example for `b` above,

```
      b/!8
2 5 6 7
```

Now for a random vector `v` to find those elements strictly between `1` and `10` we write:

```
     ((10>v)^1<v)/v<-0.1 9 12 1.2 6.3 _2 10 17 8 100
9 1.2 6.3 8
```

This is what we call *dataflow style* of programming: ELI expression organized as a *chain* of operations with the output of one feeds as an input to the next operation from right to left (it is important to put enclosing parentheses to the left operand of a function such as `^` and `/` here). It is not just a matter of making a program *succinct* but also to make its logical flow of transformations *clear*.

There are two mathematical dyadic scalar functions related to comparison (but not yielding boolean results): ***maximum*** (`~.`) and ***minimum*** (`_.`). For two real numbers `a` and `b`, `a~.b` is the large of two and `a_.b` is the smaller of the two (if one of the numbers above is truly a complex number then the operation results in a *domain error*); and these operations extend to two conforming data with one being an array as other scalar functions. For example,

```
      2~.5
5
      2~.1.2 2.3 5 0.4 11
2 2.3 5 2 11
      1.2 2.3 5 0.4 11~.!5
1.2 2.3 5 4 11
      2_.5
2
      2_.1.2 2.3 5 0.4 11
1.2 2 2 0.4 2
      1.2 2.3 5 0.4 11_.!5
1 2 3 0.4 5
```

The monadic functions denoted by `~.` and `_.` are the ***ceiling*** and ***floor*** functions. For a real number `b`, `~.b` is the smallest integer which is *larger than or equal* to `b`; and `_.b` is the largest integer which is *smaller than or equal* to `b`. For example,

```
      ~.1.2 _2.3 5 0.4 11
2 _2 5 1 11
      _.1.2 _2.3 5 0.4 11
1 _3 5 0 11
```

We have presented all scalar functions in ELI except one: the ***roll*** function `?.`, also called the *psuedo-random number generator*. For an integer `n>0`, `?.n` gives out a randomly chosen integer between `1` and `n` (in case `[]IO=1`) or between `0` and `n-1` (in case `[]IO=0`). The way this number is chosen also depends on a system variable:

```
     []RL
16807
```

(see [4]). We see that

```
      ?.100
14
      ?.100 200 300
14 152 138
```

The scalar extension here is just that the function `?.` applies a vector (or an array) is the same as it applies to each vector element individually. So for `[]IO=0`, `?.2` is either `0` or `1`. Hence, an easy way to generate a random bits-vector of length `n`, say `n=32`, is the following:

```
      []IO<-0
      ?.32#2
0 1 0 1 0 0 1 1 1 0 1 1 0 0 1 1 0 0 0 0 1 1 1 1 1 0 1 0 1 1 1 0
```

The dyadic function ***deal*** `a?.b` is not a scalar function (i.e. it is a mixed function) where `a` and `b` are positive integers with `a<=b`; it randomly picks `a` distinct integers from `1` to `b` in case `[]IO=1`, or from `0` to `b-1` in case `[]IO=0`. For example,

```
      8?.100
14 76 46 54 22 5 68 94
      26?.26
4 20 12 14 6 2 18 25 10 22 1 11 16 23 3 19 24 7 9 17 26 8 13 5 21 15
```

In particular, we see that `26?.26` is a random permutation of `!26`.

Now back to the compress function `b/a`. What if `a` is a multi-dimensional array? First, the length of `b` must equal the length of the last dimension of `a`; then `b/a` selects along last axis of `a`. For a matrix `a`, `b/a` selects to retain those columns of `a` which correspond to a `1` in `b`. For example,

```
      1 0 1 0 0 1 1/m<-5 7#!35
 1  3  6  7
 8 10 13 14
15 17 20 21
22 24 27 28
29 31 34 35
```

To select rows instead of columns we use the ***compress along the first axis*** function `b/.a` where the length of `b` must equal to the length of the first dimension of `a`:

```
      0 1 1 0 1/.m
 8  9 10 11 12 13 14
15 16 17 18 19 20 21
29 30 31 32 33 34 35
```

There is a primitive function ***expand*** `b\a` which is kind of the 'inverse' of the compress function, where `b` is boolean and the number of `1`'s in `b` equals to the vector/last dimension length of `a` or `a` is a singleton (`1=^a`); then `b\a` splits `a` (along the last axis) into a new array where those elements corresponding to a `1` in `b` are from `a` with the rest consisting of the *fill-element* of `a`'s type. The *fill-element* of *numeric* type is `0`, that of *character* type is `' '` and that of *symbol* type is `` ` ``. For example,

```
      1 0 1 0 0 1 1\'ABCD'
A B  CD
      1 0 1 0 0 1 1\8
8 0 8 0 0 8 8
```

```
      1 0 1 0 0 1 1\`ab `c `d1 `eeh
`ab ` `c ` ` `d1 `eeh
```

And the companion function ***expand along the first axis*** `b\.a` is similarly defined on a multi-dimensional array `a` as in the case of compress.

There is a dyadic primitive function membership `a?b` which for each element in `a` it asks whether the element belongs to `b` and gives `1` if yes, `0` if no. Hence, the result of `a?b` is boolean and the shape `a?b` of is the shape of `a`. For example,

```
      5?'A'
0
      w
njvfl
fnlup
afbpw
12ABC
      w?'abcdefghijk1'
0 1 0 1 0
1 0 0 0 0
1 1 1 0 0
1 0 0 0 0
```

A finite numeric (character/symbolic) *set* can obviously be represented as a vector *sv* in ELI provided that each element of *s* appears only once in *sv*; and if a predicate *P* applicable to elements of the set can be expressed as an ELI boolean expression *PB*, then the set

$$\{x \mid P(x), x \in s\}$$

can be expressed in ELI as

```
(PB(sv))/sv
```

For example, if `sv` is `!10` and P is 'x is an even number' then `PB` is `0=2|sv`:

```
(0=2|sv)/sv<-!10
2 4 6 8 10
```

and the complement of that set is `(~PB(sv))/sv`:

```
(~0=2|sv)/sv<-!10
1 3 5 7 9
```

In general, if `a` is a vector representing another set, then the *complement* of `sv` with respect to `a` is

```
(~sv?a)/sv
```

For two '*sets*' `a` and `b`, the *intersection* of the two is the following:

```
(a?b)/a
```

The difference between a *set* and a *vector* is that in a set the elements are unique while a vector can contain duplicate elements. There is a monadic primitive function ***unique*** `=a` which eliminates duplicates in vector `a`:

```
      ='njvflfnlupafbpw12ABC'
njvflupabw12ABC
      =2 1 3 1 2 5 4
2 1 3 5 4
```

For two vectors `a` and `b`, `a,b` is just a vector with `b` glued to `a`. Hence, the *union* of two sets represented by `a` and `b` is

```
      =a,b
```

The monadic form of `?` is the ***where*** function: for a boolean vector `b`, `?b` gives the positions of `1`s in `b` depending on `[]IO`:

```
      ?0 1 0 1 0 1 0 0 0 0 1 1 1 0 0 1 0 0 0 0
2 4 6 11 12 13 16
      []IO<-0
      ?0 1 0 1 0 1 0 0 0 0 1 1 1 0 0 1 0 0 0 0
1 3 5 10 11 12 15
```

While the dyadic function `a?v` indicates which elements of `a` belongs to `v`, for a vector `v`, the dyadic function ***index of*** `v!a` gives more information: the *position* (*index*) *of* each element in `a` first appears in `v` and for elements of `a` which is not in `v` the corresponding result is `1+#v` (or `#v` if `a[]IO=0`) indicating the element is out of bound of `v`. The shape of `v!a` is the shape of `a` and we must have `v` and `a` of the same type. For example

```
      'abcdefghijklmnopqrstuvwxyz'!'i like to see you'
9 27 12 9 11 5 27 20 15 27 19 5 5 27 25 15 21
      a<-5 7#!35
      a
 1  2  3  4  5  6  7
 8  9 10 11 12 13 14
15 16 17 18 19 20 21
22 23 24 25 26 27 28
29 30 31 32 33 34 35
      #v<-2 3 5 7 11 13 17 19 23 29 31 37 39
13
      v!a
14  1  2 14  3 14  4
14 14 14  5 14  6 14
14 14  7 14  8 14 14
14  9 14 14 14 14 14
10 14 11 14 14 14 14
```

1.6 Array Indexing, Indexed Assignment and taking Sections

We have already seen that elements of an array can be selected by the compress function. More conventionally, elements of an array can be *selected by their positions* in an array, i.e. indexing as in any other high-level programming languages. *Indexing* in ELI depends on `[]IO` , the *index origin*, which we assume to be `1` here unless specified otherwise.

```
      n<-55 47 77 93 31 96 7 58 45 16
      n[1]
55
      n[3]
77
      n[10]
16
```

```
      []IO<-0
      n[1]
47
      n[3]
93
      n[10]
index error
      n[10]
        ^
```

We can see that for `[]IO=0` it is just like in C, and if the index is out of bound it would result in an `index error`. More importantly, an array can be *indexed* by a *vector*: as long as each element of that vector is within index bound,

```
      n[!3]
55 47 77
      n[5+!5]          (or n[6 7 8 9 10])
96 7 58 45 16
```

and one can scramble order of indexing, have repetition in indexes and even a change of shape in indexing set:

```
      n[5 2 3 1]
31 47 77 55
      n[6 6 5]
96 96 31
      m<-3 4#!10
      m
1  2 3 4
5  6 7 8
9 10 1 2
      n[m]
55 47 77 93
31 96  7 58
45 16 55 47
```

For a vector `n` and an array `I` we always have

```
#n[I] ←→ #I
```

where each element of `I` must be an integer from `!#n`.

For a matrix (or an array of higher dimension), ordinary scalar indexing as well as vector indexing are all allowed with indexing coordinates to each dimension separated by a '`;`':

```
      c
ABCD
EFGH
IJKL
      c[2;4]
H
      c[2 3;4]
HL
      c[2 3;1 4]
EH
IL
      c[;4]
DHL
      c[1 2;]
ABCD
EFGH
```

where an *empty expression* before or after a ';' indicates that all elements in the corresponding axis would be taken. In general, if `a` is an k-dimensional array, then an ***index expression***

$$I \leftrightarrow I_1;I_2;\ldots;I_k$$

is *legal* for `a` provided that each I_j is either empty or is an integer expression whose value (or value of its elements in case of an array) lies within `!(#a)[j]`. Each I_j is called a *component* of the index expression `I`. For such an index expression, the shape of `a[I]` is the Cartesian product of shapes of I_j's. For example,

```
      a<-2 3 4#!24
      a
 1  2  3  4
 5  6  7  8
 9 10 11 12

13 14 15 16
17 18 19 20
21 22 23 24
      a[1;2 3;1 2 4]
5  6  8
9 10 12
      a[;1 3 2;2 4]
 2  4
10 12
 6  8

14 16
22 24
18 20
```

What happen if one of the element in I_j is not in `!(#a)[j]`? Simple, the system will respond with a message

```
index error
```

and execution stops.

We have already introduced ***assignment***

```
Av<-expre
```

informally in section 1.2, where `Av` is the name of an ELI variable and `expre` is any valid ELI expression, i.e. its evaluation (to be explained in detail in the next chapter) must result in a well-defined value. The expression can involve several function applications or can simply be a literal value or another variable which already has a value. In particular, if we want to assign a value such as `!0` to several variables `B`, `C`, `D`, `E` we can write in one line

```
B<-C<-D<-E<-!0
```

The result of the assignment is that the variable on the left of `<-`will have the value of the expression on the right of `<-`. Unlike ***typed*** languages like FORTRAN, C or Java, there is no restrictions what so ever on the variable or the expression of either sides of `<-`, i.e. any legal expression can be assigned to any variable at any time. In FORTRAN or C, a variable must be declared implicitly or explicitly to be of certain type, say, integer, floating-point or character, and only expressions of corresponding type can be assigned to a particular variable. In ELI, a variable can change

from a floating-point matrix to a character vector or scalar symbol at any time (though it is not a good programming practice to change the type of a variable at a whim). This saves user the hassle of variable declaration. More importantly, it let you program with variables whose dimensions or sizes can change during program execution.

Besides the simple assignment we discussed above, ELI provides another kind of assignment, called ***indexed assignment*** of the following form:

```
Arv[index_expre]<-expres
```

where `Arv` must be a variable whose current value is an array, `index_expre` is an index expression of `Arv` as we have discussed earlier on indexing an array (a vector, matrix or multi-dimensional array). Unlike in the case of simple assignment, there are restrictions on `expres`: First, must be of the same storage-type as that of `Arv`, i.e. they must be both numeric, or both character or both symbolic. Second, `expres` must be either a scalar or of the same shape as that of `index_expre`. The effect of the indexed assignment is the replacement of the values of the elements of the array selected by `index_expre` as in indexing by the corresponding elements of `expres` (in case there are repeat elements in `index_expre` then the later ones in `expres` overwrite earlier ones). If `expres` is a scalar, then every selected element of the array is replaced by that scalar. For example,

```
      av<-!12
      av[2 3]<-12 13

      av
1 12 13 4 5 6 7 8 9 10 11 12
      av[2*!6]<-0
      av
1 0 13 0 5 0 7 0 9 0 11 0
      a<-2 3 4#!24
      a[1;2 3;3 4]<-2 2#!4
      a
 1  2  3  4
 5  6  1  2
 9 10  3  4

13 14 15 16
17 18 19 20
21 22 23 24
      a[2;;1 3 4]<-1
      a
1  2 3 4
5  6 1 2
9 10 3 4

1 14 1 1
1 18 1 1
1 22 1 1
```

Combined with the monadic function *where* `?`, we have a very convenient way to replace certain elements in a vector such as `'A'` by `'a'` in `ch` and all negative numbers in `w` by `0`:

```
      ch
A book named 'ABC'
      ch[?ch='A']<-'a'
      ch
a book named 'aBC'
      w
_1.5 2 3.1 _0.3 10 9 _3
```

```
      w[?w<0]<-0
      w
0 2 3.1 0 10 9 0
```

We note that indexed assignment is the only kind of assignment which allows expressions other than a variable name appear on the left side of `<-` in ELI.

Many times, one would like to take some segment of a vector `v` (section of an array) and there is a dyadic ***take*** function `^.` to do just that: for a vector `v` the function `n^.v`, where `n` must be an integer, can be illustrated by the following examples:

```
      3^.1 2 3 4 5 6 7
1 2 3
      _3^.1 2 3 4 5 6 7
5 6 7
      9^.1 2 3 4 5 6 7
1 2 3 4 5 6 7 0 0
```

i.e. for `n>0` (resp. `n<0`) `n^.v` takes the first (resp. last) elements of `v`, and if `|n|>#v`, additional slots are filled by a *typical element* of the type of `v`. If the right argument of *take* is a matrix `a`, the left argument must be a vector `n2` of length 2; `n2[1]` indicates the number of rows of `a` to take along the first axis and `n2[2]` indicates the number of colunms to take along the second axis (and this principle is extended to multi-dimensional right argument, i.e. the length of the left argument vector must equal to the rank of the right argument). For example,

```
      A
abcd
efgh
ijkl
abcd
      2 3^.A
abc
efg
      _1 2^.A
ab
```

We note the *take* of a vector always ends in a vector. Hence, for a vector `a` there is a subtle difference between `a[1]`, which is a scalar since its index, `1`, is a scalar, the first element of `a`, and `1^.a` which is a one element vector made out of the first element of (`##a[1]` is `0` but `##1^.a` is `1`). In fact, for a vector `a`, we have a monadic primitive function ***first*** `^.a` for `a[1]`.

```
      a<-11 3.2 9 10
      a[1]
11
      ^.a
11
      ##^.a
0
      1^.a
11
      ##1^.a
1
```

There is a dyadic primitive function called ***drop*** (`n!.a`) which is the opposite of *take*: it drops the first, if `n>0` (resp. last, if `n<0`) elements of `a` and returns the rest, assuming `a` is a vector:

```
      1!.a
3.2 9 10
      3!.a
10
      _2!.a
11 3.2
      5!.a

      ##5!.a
1
      #5!.a
0
      0!.a
11 3.2 9 10
      #0^.a
0
```

We note that if `n>=#a` the result is an empty vector. We also see that `0` *drop* of `a` returns `a` while `0` *take* of `a` is an *empty* vector. This rule for drop on vectors can easily be extended to multi-dimensional right arguments similar as in the case of the take function, i.e. the first element of left argument applies to the first axis of the right argument and so on. For example, for the matrix `A` appeared previously,

```
      1 2!.A
gh
kl
      _1 0!.A
abcd
efgh
```

1.7 Array Transformations

ELI provides many primitive *mixed* functions to transform whole arrays or make a new array out of one or two argument arrays. In fact, we have already seen such examples in the *take* and *drop* functions in the previous section that make new arrays out of sections of an argument array. We shall start with the monadic ***ravel*** function `,a`, which turns its right argument `a` into a vector consisting of the same elements as that of the original array in *raveled* order:

```
      m
1  2  3  4
5  6  7  8
9 10 11 12
      ,m
1 2 3 4 5 6 7 8 9 10 11 12
```

If `a` is a vector then `,a` ←→ `a` (invariant). If `a` is a scalar, however, the result is an one element vector.

```
      a<-2
      b<-,a
      b
2
      #b
1
      #a
          (* an empty vector *)
```

The dyadic function `a,b` is called ***catenate*** which '*glues*' its two arguments `a` and `b`. For `a` which is either a scalar of a vector and `b` which is of similar type as that of `a`, we have

```
      'A','CE'
ACE
      a<-1 3 5
      b<-2 4 6 8
      a,b
1 3 5 2 4 6 8
      s<-3#`abc
      s1<-`dkb
      s,s1
`abc `abc `abc `dkb
```

For more general case, `a,b` (called `a` *catenate* `b`) concatenates two arrays along the last axis. For example, with `m` above and `n` below, are two matrices with equal length first axis, we have:

```
      n
0 0 0
0 0 0
0 0 0
      m,n
1  2  3  4 0 0 0
5  6  7  8 0 0 0
9 10 11 12 0 0 0
      n,m
0 0 0 1  2  3  4
0 0 0 5  6  7  8
0 0 0 9 10 11 12
```

One of the argument to *catenate* can be a scalar or a vector while the other is a matrix (in the 2nd case, the length of the vector must equal to the length of the 1st dimension of the array). For example,

```
      m,30
1  2  3  4 30
5  6  7  8 30
9 10 11 12 30
      m,3 33 99
1  2  3  4  3
5  6  7  8 33
9 10 11 12 99
```

What about *catenate*, i.e. glue, two arrays along the first axis? Indeed, ELI has a function ***catenate*** *along the first axis*, `a,.b`, to do so with similar requirements on its arguments as in the case of `a,b`:

```
      n1
0 0 0 0
0 0 0 0
      m,.n1
1  2  3  4
5  6  7  8
9 10 11 12
0  0  0  0
0  0  0  0
      m,.30
 1  2  3  4
 5  6  7  8
 9 10 11 12
```

```
30 30 30 30
      m,.3 33 66 99
1  2  3  4
5  6  7  8
9 10 11 12
3 33 66 99
```

We note that `a,.b` can also be specified as `a,[1]b` (for `[]IO=1` or `a,[0]b` for `[]IO=0`). This `[1]`is called an *axis specification* and can be applied to an axis other than the first or the last (see [4]).

For two arrays `a` and `b` of the same shape, we can specify a *fractional* number `f` in the *axis specification* to get a new array `a,[f]b` which joins the two argument arrays along a new axis whose relative position is specified by `f`. This function is called laminate. For example,

```
      c
1 2 3 4 5
      d
0 0 0 0 0
      c,[0.5]d
1 2 3 4 5
0 0 0 0 0
      c,[1.5]d
1 0
2 0
3 0
4 0
5 0
```

We see that if `f<1` (the first dimension here since `c` is a vector) then the newly created axis is the 1^{st} axis; if `f>1` (the last axis) then the newly created axis is the *last* axis.

If we want to generate a vector `v` of `n` integers start from `s` and `p` apart, we set `[]IO<-0` and write

```
v<-s+p*!n
```

Now, suppose we like to list a short table of a regular sequence of 10 temperatures in Celsius starting at -5 and 3 degrees apart with their corresponding Fahrenheit temperatures, we do the following:

```
      []IO<-0
      _5+3*!10
_5 _2 1 4 7 10 13 16 19 22
      c<-_5+3*!10
      32+1.8*c<-_5+3*!10
23 28.4 33.8 39.2 44.6 50 55.4 60.8 66.2 71.6
      c,[0.5]32+1.8*c<-_5+3*!10
_5 23
_2 28.4
 1 33.8
 4 39.2
 7 44.6
10 50
13 55.4
16 60.8
19 66.2
22 71.6
```

In comparison with the C program to do conversion in [11], we see that C relies on loops to do computation on array elements whereas in ELI arrays are ***first class citizens*** manipulated by language primitives operating on arrays to eliminate the need of looping in many cases. This is what we mean by ***programming with arrays***.

The monadic function ***reverse*** `$a` reverses the elements of an array `a` along its last axis, or any other axis specified by an axis operator `$[f]a`. For example (`[]IO=0`),

```
      $!5
4 3 2 1 0
      $m
 4  3  2 1
 8  7  6 5
12 11 10 9
      $[0]m
9 10 11 12
5  6  7  8
1  2  3  4
```

In particular, for the temperature conversion code above, we may prefer to list from high temperatures to lower ones:

```
      c,[0.5]32+1.8*c<-$_5+3*!10
22 71.6
19 66.2
16 60.8
13 55.4
10 50
 7 44.6
 4 39.2
 1 33.8
_2 28.4
_5 23
```

We note that *reverse* of `a` along the *first axis* can simply be written as `$.a`.

The dyadic function ***rotate*** `n$a` rotates the elements in `a` with the amount specified by `n`, which must be integral and its length must equal to the rank of `a`, in such a way that a positive integer indicates the amount of elements moved from left to right (or top to bottom in case of rotating along first axis) and a negative integer indicates a rotation in reverse direction.

```
      v
1 2 3 4 5
      2$v
3 4 5 1 2
      _1$v
5 1 2 3 4
```

For a multi-dimensional array `a` as the right argument of `n$a` (resp. `n$.a`), the shape of `n` is required to be

`#n` ←→ `_1!.#a` (resp. `#n` ←→ `1!.#a`)

i.e. drops the part of shape of `a` representing the dimension of the axis it is rotating about; and each element in `n` indicates how the corresponding row (resp. column) in `a` is to be rotated. For example,

```
      A
abcd
```

```
efgh
ijkl
      1 2 3$A
bcda
ghef
lijk
      2 0 1 3$.A
ibgd
afkh
ejcl
```

For a multi-dimensional array `a` such as a matrix, the left argument `n` to the rotate function `n$a` can be a scalar. In this case `n` is expanded to `(##a)#n`, i.e. the rows or columns of `a` will be rotated the same amount `n`. For example,

```
      1$A
bcda
fghe
jkli
      _1$A
dabc
hefg
lijk
      1$.A
efgh
ijkl
abcd
      _1$.A
ijkl
abcd
efgh
```

The monadic function ***transpose*** `&.a` reverses the order of axis of `a`. In case of a matrix, it just exchanges the first axis with the last axis of the argument. For `A` above, we have

```
      &.A
aei
bfj
cgk
dhl
      m2<-2 3 4#!24
      &.m2
 1 13
 2 14
 3 15

 4 16
 5 17
 6 18

 7 19
 8 20
 9 21

10 22
11 23
12 24
```

The dyadic function ***general transpose*** `n&.a` where `n` is a vector of length equal to the rank of `a` with elements coming from shape vector of `a`. We will not go further into details but just note that `1 1&.a` is taking the diagonal of `a`:

```
     1 1&.A
afk
```

1.8 Operators and Derived Functions

Besides an abundance of primitive functions (in fact, we have not presented all primitive functions in ELI yet), what makes APL/ELI so powerful are the *operators* it provides. An ***operator*** in ELI applies to one or two primitive scalar functions to produce a new function, called a ***derived function***. We have already encountered the reduction operator `/` applied to the add function `+`, `+/`, earlier in sect. 1.6, and we can also regard the axis modification `[x]` of compression and expansion as an operator on the compress and expand functions (these are not scalar functions). The regular operators are the following:

The ***reduction operator*** denoted by **`/`** takes a left function argument `f`, which must be a dyadic scalar function, to produce a new monadic *derived function* `f/`. For a vector `v` whose elements are $v_1, v_2, \ldots v_n$

$f/v \leftrightarrow v_1 f v_2 f \ldots f v_n$

We have already seen earlier that `+/` is the *summation* function, and it is easy to see that `*/` is the *product* function. Other useful derived functions from reduction are the *maximum* `~./` and *minimum* `_./` functions:

```
      ~./3.2 4 8 0.2 9
9
      _./3.2 4 8 0.2 9
0.2
```

The result of `f/A` for a vector `A` is always a scalar; see [4] for cases of an empty vector `V` (`+/V` is `0` and `*/V` is `1`). In general, we have

`#f/A` ↔ `_1!.#A` (`#f/.A` ↔ `1!.#A`)

In other words, `f/A` always results in an array with a rank 1 less than the rank of `A`. For an array `A`, the reduction always applies along the last axis. To do reduction along the first axis, we use the operator, ***reduce along 1st axis*** `f/.A`. For example,

```
      m
1  2  3  4
5  6  7  8
9 10 11 12
      +/m
10 26 42
      +/.m
15 18 21 24
```

The ***scan*** operator denoted by `\` also applies to a left argument function `f`, which must be a dyadic primitive scalar function, to produce a monadic derived function `f\`. If `v` is $v_1, v_2, \ldots v_n$, a vector of `n` elements then the k-th element of `f\v` is `f/wk`, where `wk` is the vector $v_1, v_2, \ldots v_k$ (i.e. `k^.v`). For example,

```
      a
1 2 3 4 5
      +\a
1 3 6 10 15
      *\a
1 2 6 24 120
      ^\1 1 0 1 0
1 1 0 0 0
      &\0 0 1 0 0
0 0 1 1 1
```

We see that `+\` is the *partial sum* function and `*\` is the *partial product* function. When *and-scan* `^\` applies to a boolean vector `b` it produces a new boolean vector whose elements becomes `0` once a `0` element is encountered in `b`. For the *or-scan* `&\`, it produces a boolean vector whose element starts to be `1` once a `1` is encountered.

Let `c` be the coefficient vector of a polynomial, and we assume `[]IO=0`, i.e. `c` represents the following polynomial of degree `n`:

`c[0]+c[1]x+c[2]x*x+…+c[n]x`^{n}

If we want to evaluate this polynomial at point `p`, we do

```
+/c*1,*\(_1+#c)#p
```

We note that the expression to the right of scan is a vector of `n(=(#c)-1)` `p`'s, and the scan results in a vector whose successive elements are increasing powers of `p`.

The application of `f\` to general arrays is similar to that of `f/`, i.e. along the last axis; and there is a companion operator `f\.`, ***scan along the first axis***. For example,

```
      +\m
1  3  6 10
5 11 18 26
9 19 30 42
      +\.m
 1  2  3  4
 6  8 10 12
15 18 21 24
```

The ***outer product*** *operator* `.:` applies to a *right argument function* `f` which must be a dyadic scalar function and produces a dyadic *derived function* `.:f`. For vectors `v` and `w`, `v.:f w` is a matrix `M` whose individual element is defined as follows:

`M[i;j]` ←→ `v[i]f w[j]`

For example,

```
      a0<-1 2 3
      a
1 2 3 4 5
      a0.:*a
1 2 3  4  5
2 4 6  8 10
3 6 9 12 15
      a0.:<a
```

```
0 1 1 1 1
0 0 1 1 1
0 0 0 1 1
```

In general, for arrays `A` and `B`, `A.:f B` is a Cartesian product of `A` and `B`, and we have

```
#A.:f B ←→ (#A),#B
```

i.e. the shape of the result of applying the derived function of an outer product to two arguments is the catenation of the shapes of the argument arrays, and each element of the result is from a pairwise application of the argument function `f` to corresponding elements in the argument arrays.

Outer product is often used to construct a table for a range of various input values. For example, if one deposits `1000` dollars at 3% annual interest rate, after one year one would have `1000*1.03` and `1000*1.03*1.03` in two years. During ten years period, the money grows as

```
1000*1.03*.1 2 3 4 5 6 7 8 9 10
```

To see how 1000 dollars will grow under 3%, 5% and 8% of annual interests in 10 years, we do

```
      1000 *1.03 1.05 1.08 .:*. !10
1030  1060.9 1092.727 1125.5088 1159.2741 1194.0523 1229.8739 1266.7701 1304.7732 1343.9164
1050  1102.5 1157.625 1215.5063 1276.2816 1340.0956 1407.1004 1477.4554 1551.3282 1628.8946
1080  1166.4 1259.712 1360.489  1469.3281 1586.8743 1713.8243 1850.9302 1999.0046 2158.925
```

The ***inner product*** *operator* denoted by ':' takes two dyadic primitive scalar function arguments `f` and `g` to produce a *dyadic derived function* `f:g`, which for two vectors `v` and `w` of equal length is defined as follows:

```
v f:g w   ←→   f/v g w
```

For example,

```
      v<-2 5 1 4
      w<-1 9 6 7
      v~.:+w    //ELI parser scans from left to right, so it is interpreted as v(~.):+w
14
      v+:~.w
24
```

The (derived function of) inner product `f:g` extends to arrays `M` and `N` along the last axis `M` of and the first axis of `N`. Hence, we must have `(_1^.#M)=1^.#N`, and the shape of the result of the application of the derived function is

```
#M f:g N  ←→   (_1!.#M),1!.#N
```

For example,

```
      m
1  2  3  4
5  6  7  8
9 10 11 12
      n
1 2 3
4 5 6
      m+:*n
```

```
length error
      m+:*n
       ^
      n+:*m
38 44  50  56
83 98 113 128
      n*:+m
168  312  504  750
750 1056 1428 1872
```

We can see that for two numeric matrices `n` and `m`, `n+:*m` is the familiar *matrix product* of `n` and `m`. But other inner products are also useful. Suppose `T` is a 10 by 8 name-table and `n1` is a name (of 8 characters padded with blanks):

```
      T
Anthony
Backus
Cocke
Codd
Donath
Eastman
Fagin
Gabriel
Harrison
Lucus
      n1<-'Cocke   '
      T^:=n1
0 0 1 0 0 0 0 0 0 0
      ?T^:=n1
3
```

That means `n1` is the third name in the table as only the 3rd row of `T` matches every character in `n1`.

1.9 Lists and Operations on Lists

A fundamental data structure in ELI which is not in classical APL [9] is the *list*: a ***list*** is a group of *item*s, each of which can be a *scalar*, an *array*, or another *list*, separated by '`;`':

```
   l<-(1 2;`abc;'vb')
   l
<1 2
<`abc
<vb
```

We see that items in a list, unlike that in an array, can be of different types or even of different structures. Hence, in ELI we use lists to organize *non-homogeneous* data. What can we do on lists? First, we can apply *shape* function `#` (or *count* `^`) to a list to get the number of items in it:

```
     #l
3
```

reshape **does not** work for lists, but a special case of *reshape* works for allocating a list of `n` *empty* items (denoted by the *underline* symbol _) or get a 0 item list:

```
     #Ln<-3#_
3
```

```
     l0<-0#_
     #l0
0
```

A *list* can be indexed like a vector. Most importantly, a list `L` can be assigned to a *group* of variables all at once, where the number of variables is equal to `#L`. A *list* can be entered with *items* containing expressions or another list:

```
      l[1 3]
<1 2
<vb
      (n;s;c)<-l
      n
1 2
      s
`abc
      c
vb
      a<-2 3 4
      (a+1;$'avc';;!10)
<3 4 5
<cva
<
<1 2 3 4 5 6 7 8 9 10
```

A scalar `s` or an array `a` is **not** a list, to make an one item list of it, we employ the monadic primitive function ***enclose*** `<.`:

```
       #L<-<.2 4#!12
1
       L
<1  2  3  4
 5  6  7  8
```

take, *first*, *drop* and *catenate* work on lists the same way as they work on vectors (see [4] sect.2.1). In particular, for a one-item list `L1` first `^.L1` serves as *disclose* of `L1`, i.e. it turns `L1` back into a scalar or an array:

```
       ^.L
1  2  3  4
5  6  7  8
       #^.L
2 4
```

The ***each*** *operator* `"` operates on lists, or a scalars/array and a list. For a *monadic* function *f*, and a list `L`, `f"L` is evaluated by applying `f` to each component item `L[j]` of `L` to result in a list `Z` of the same structure as that of `L`, i.e. `#f"L` is of the same length `#L` and each element of `Z[j]` is `fL[j]` which must be well-defined for all `j`.

```
      #"l
<2
<
<2
      3#"l
<1 2 1
<`abc `abc `abc
<vbv
      ln<-(!5;!10)
      ln
<1 2 3 4 5
```

```
<1 2 3 4 5 6 7 8 9 10
      +/"ln
<15
<55
```

We see that unlike in operators for functions on arrays in the previous section, the argument *f* to the *each* operator is not restricted to scalar primitive functions; it can be a mixed function, a derived function or even a defined function (see next section).

For a dyadic function *g*, the ***derived function*** *g*″ is also a dyadic function: for two lists L and R, *g*″ is defined as

(L *g*″ R)[i] ←→ L[i] *g* R[i]

for each i in !#L, where we must have (#L)=#R, and as in monadic case, *g* can be a mixed function, derived function or a defined function, but for each i, L[i] *g* R[i] must be well-defined; either L or R can be a scalar or a one-item list by replacing L[i] (resp. R[i]) with L (resp. R) in the formula above. For example,

```
      (1;2;3)$"(!5;!8;!10)
<2 3 4 5 1
<3 4 5 6 7 8 1 2
<4 5 6 7 8 9 10 1 2 3
      1+"(!5;!8;!10)
<2 3 4 5 6
<2 3 4 5 6 7 8 9
<2 3 4 5 6 7 8 9 10 11
      (2 3;3 4)#"<.!10
<1 2 3
 4 5 6
<1  2 3 4
 5  6 7 8
 9 10 1 2
```

Note that in the last example it is necessary to *enclose* !10 to make it a one-element list, otherwise, it will result in a *length error*. See section 2.3 in [4] for more examples of *each* operator.

Suppose we have a linear algebra class and students taking the course are from three departments: mathematics, computer science and engineering; and we enter the final test scores of students by departments as follows:

```
      score<-(math;cs;eng)<-(70 85 72 88 69 96 58 100 75 60;87 77 64 50 92 80 73 96;84 73 66 97
54 79 62 81 70)
      math
70 85 72 88 69 96 58 100 75 60
      cs
87 77 64 50 92 80 73 96
      eng
84 73 66 97 54 79 62 81 70
      score
<70 85 72 88 69 96 58 100 75 60
<87 77 64 50 92 80 73 96
<84 73 66 97 54 79 62 81 70
```

First, we notice that the list notation allows us to assign *multiple variables* at once. Now we would like to count the number of students from each department, the total score and the average score as well as the maximum/minimum score for students from each department.

```
      #"score
<10
<8
<9
```

```
       +/"score
<773
<619
<666
      (+/"score)%"#"score
<77.3
<77.375
<74
      ~./"score
<100
<96
<97
      _./"score
<58
<50
<54
```

In fact, we can write a short function (see the following section 2.1) which gives the average of a numeric vector

```
      {avg0:(+/x)%^x}
avg0
      avg0 80 90 76
82
```

and apply *each* to it to get the same result

```
      avg0"score
<77.3
<77.375
<74
```

What if we just want to count, calculate the total, average, maximum/minimum of the whole class? Easy, there is a monadic function raze (,.) which turns a homogeneous list (i.e. the list elements are of the same type) into a vector, and then we can simply apply appropriate functions to that vector:

```
      ,.score
70 85 72 88 69 96 58 100 75 60 87 77 64 50 92 80 73 96 84 73 66 97 54 79 62 81 70
      #,.score
27
      +/,.score
2058
      avg0 ,.score
76.22222222
      ~./,.score
100
      _./,.score
50
```

Now suppose the test scores of the whole linear algebra class was entered as a vector in a random fashion but an accompanying vector of symbols indicates which department the student with that score comes from:

```
     mscore<-88 66 64 92 96 85 84 81 87 54 70 77 80 72 96 62 100 73 75 70 60 58 79 69 50 73 97
     dps
`math `eng `cs `cs `math `math `eng `eng `cs `eng `math `cs `cs `math `cs `eng `math `eng `math
    `eng `math `math `eng `math `cs `cs `eng
```

Can we reconstruct a list of scores by departments to get a departmental view of the performance of students in the linear algebra class? Indeed, it is quite simple to do so. First, there is primitive monadic function ***unique*** (=) in ELI which when applying to a vector `v` will yield unique elements of `v`; so

```
      =dps
`math `eng `cs
```

Second, there is primitive monadic function grouping (`>.`) in ELI which when applying to a vector `v` will yield a list of length the number of unique elements in `v` and each item of that list consists of the indices of elements in `v` equal to a particular elements in `=v` as follows

```
     >.dps
<1 5 6 11 14 17 19 21 22 24
<2 7 8 10 16 18 20 23 27
<3 4 9 12 13 15 25 26
```

Third, for a vector `v` and a list `I` such that sub-elements of `I` are valid indices of `v`, then `v[I]` is a list of the same structure as that of `I` with corresponding elements being the indexed elements of `v`; so

```
      mscore[>.dps]
<88 96 85 70 72 100 75 60 58 69
<66 84 81 54 62 73 70 79 97
<64 92 87 77 80 96 50 73
```

Thus, we reconstructed test scores of the class according to the departments the students are from.

2. Defined Functions, Control Structures and Files

2.1 Defined Functions, Short-Form and Order of Evaluation

We have seen many examples of ELI expressions, now we would like to find a way to save these codes for reuse: we put the code into a *defined function* by switching to the ***edit mode*** of ELI. The edit mode is triggered by a ***del***-symbol `@.` followed by a function name `fn`, or a function name together with formal parameter name(s):

```
@.fn
```

and end the edit mode by entering a matching `@.`. A *parameter* is a variable whose *value* is assigned at the time of function call from that of an *actual parameter*; the rule to form a *function name* is the same as that for a variable name (see sect. 1.2). Just like primitive functions, a defined function can be monadic or dyadic, i.e. take a right argument only, or take a left argument as well as a right argument. However, a defined function can take no

argument and it is then called a ***niladic*** function. And unlike a primitive function which always gives a result, a defined function can give a result or return no result. These distinctions of a defined function are all evident in the first line (line [0]) of a defined function which is of the following form

`function-header <;local-variable-list>`

`function-header` must be one of the six forms below:

type	valence	have result	no result
`niladic`	**`0`**	`R<-FN`	`FN`
`monadic`	**`1`**	`R<-FN B`	`FN B`
`dyadic`	**`2`**	`R<-A FN B`	`A FN B`

where ***valence*** is the number of arguments which the function takes, `R` is the name of the returned result, `FN` is the name of the function, `A` is the name of the *left argument* and `B` is the name of the *right argument* (you can, of course, give different names to each one of them). The arguments can also be arrays and lists, not just scalars. One can use an explicit list notation on the right argument to effectively input more than 2 arguments. For example, after type

```
@.z<-a try (b;c)
```

then type two lines `z<-a*b+c` and `@.` in the *function edit panel* to get a defined function `try`. We then test it and display its definition:

```
      3 try (5;6)
33
      {try}
---------try--------
[0]   z<-a try (b;c)
[1]   z<-a*b+c
--------------------
```

We note that use `{fn}` to display function `fn` only works after `fn` has been executed. To see a function text right after it is defined, we can use the system function `[]CR` which returns the character matrix of pure function text:

```
      []CR 'try'
z<-a try (b;c)
z<-a*b+c
```

The **`local-variable-list`** in the function header-line is *optional*; it is required only if you wish to ***localize*** a group of variables. The list starts with a '`;`', and variable names in the list are separated by '`;`'. If a variable named `V1` is *localized* in a function `AF`, then `V1` must first be assigned a value in `AF` before it can be used in `AF`. The parameter(s) and the result, if any, of a defined function are *automatically localized*. We just point out that unlike in C where a variable is either *local* (to a procedure) or *global*, a variable in an ELI function `fn` can be local to some function `fn1` which calls `fn` directly or through a *chain* of *calling* functions. In other words, the ***scoping rule*** in ELI is ***dynamic***, not ***static*** as in PASCAL (see sect. 3.1 in [4] for a more detailed explanation of *shading*).

When a function

```
@.z<-a fn b;…
 …
 …
@.
```

is called by `v fn w`, `a` gets the value of `v` and `b` gets the value of `w` before the code body of function `fn` starts to execute. The *parameter passing* mechanism in ELI is ***by-value***, i.e. values of `v` and `w` would not be changed when the function `fn` returns no matter what happen to `a` and `b` during the execution of `fn`.

We remark that even though it looks like a defined function can take only two parameters, by using a *list* as the *right* parameter we can actually pass in *multiple* parameters; for example, with a function head of the following form

```
@.z<-a fn (b1;b2;b3);…
```

we can pass 4 parameters.

From the previous chapter, we already have a good feel as how an ELI expression is evaluated. State more formally: a *line of* ELI *code* `L`, also called a *simple* ELI *statement*, is a group of constants and variables interspersed with primitive functions, derived functions and defined functions, and possibly with parentheses. We note here that the assignment `<-` is also regarded as a primitive function. To ***evaluate*** `L`, one starts with the right most item `r` in `L` which must be either a *literal constant*, a *variable name* or a *niladic function name* with a result, in case it is a '`)`' we move to the left to *evaluate* the sub-line up to a matching '`(`' to get a value. In case it is a variable name, ELI fetches its value; in case of a function name, ELI executes that function to get a resulting value. Execution of `L` fails if either variable value is missing of function runs into trouble; otherwise we have a value `v`. Either we reach the end of `L` or we proceeds to the item `l` next to the left in `L`; `l` must be i) a primitive or derived function symbol `pf` or ii) a defined function name `fn`. In case of i) and `pf` is ambivalent, we check to see whether there is an item on the left of `pf` which will result in a value `w` and decides `pf` to be dyadic. This includes the case of a '`)`' for which we call ***evaluate*** recursively to the sub-line up to a matching '`(`' to get a value `w`. We either get a new value `w pf v`, or `pf v` if there is no item to the left of `pf`. In case of a defined function `fn` then whether we proceed to get `w fn v` or `fn v` is determined by the valence of `fn`. In any case we replace `v` by the new value, and either we continue to evaluate *leftward* or we reach the *end* of evaluation of `L`. If at any step, we run into problem such as a missing value (`value error`) or mismatched arguments (`length error` or `doamin error`), the execution stops with an ***error message***. A ***successful*** *evaluation* of `L` always results in a *value* `V`. For example,

```
ln2blk[n]<-fblkx<-fblkx+newblk<-newblk&nx<-n?lx
```

is an ELI statement. The rule of ***equal precedence*** among primitive functions is almost necessary in view of the abundance of primitive functions in ELI. This simple rule also enables a natural concatenation of many operations into one line, thus contributes significantly to the ***succinctness*** of ELI programs. Parsimony of symbols is a form of

economy in the cleric aspect of preparing a program which also reduces the chance of committing trivial error.

Quite often, one would like to write a simple function of one or a few short lines. ELI provides a ***short-form*** *function definition* facility for that convenience as follows

```
{fnam: …}
```

where `fnam` is the name of a function, and either `z` or the last expression is the result of the function while `x` is the right argument, and `y` is the left argument if present; all other variables are assumed to be local. All statements must be a *simple* or *if* statement (see sect. 2.3); two statements can be separated by ';' in one line. Moreover, statements in short-form definition **cannot access** any global variable other than *system variables*. In addition, comment line(s) must be outside of the function body `{..}`, and a short-form definition of a defined function can be entered in *execution mode* of ELI, i.e. interactively. For example

```
//average of a numeric vector
```

```
{avg: (+/x)%^x}
avg   //response to the definition
      avg 4 5 0.8 10 4.7
4.9
```

The average of a vector is the sum divided by the number of elements in it. The reason we used `^` instead of `#` is that `x` could be one number only, i.e. a scalar.

Let us consider the problem of finding the greatest common divisor of two integers `x` and `y`. The function `gcd2` in short-form below solves that problem:

```
      {gcd2:_1^.(0=(a|x)+a|y)/a<-!x_.y}
gcd2
      18 gcd2 8
2
       6 gcd2 27
3
```

This is a rather naive algorithm: take the smaller of the two numbers, generate a vector `a` from `1` up to that number and select those elements of `a` which can divide evenly into (using modulo function `|`) both `x` and `y`, and take the last one from the resulting vector. While this function is quick to code in ELI and avoids the complication of *if-then-else* and *iterations*, for very large `x` and `y`, the function is quite inefficient as it incurs unnecessary computations. But it has one advantage, i.e. it can turn into a vector form effortlessly: instead of finding the greatest common divisor of two numbers, suppose we want to find the greatest common divisor of `x` which is a vector of positive integers. We have

```
      {gcd:_1^.(^/0=a.:|x)/a<-!_./x}
gcd
      gcd 18 8
2
      gcd 45 15 20
5
      gcd 9 21 60 18 24
3
```

The algorithm is the same as in `gcd2`. We just used derived functions on a vector and a matrix produced by an outer product. This example illustrates the *essence* of ***array-oriented programming*** in ELI, i.e. utilizing array operations whenever possible, instead of picking scalar elements and doing sequential iterations. While ELI is not strictly a functional language in the sense of Haskell, its programming style is quite functional, i.e. for many programming tasks, an ELI program needs very few, or no state variables to accomplish a job; it is as mathematical as a program can be formulated. Nevertheless, as we pointed out earlier, an array-oriented solution of a problem may be wasteful and problematic for very large input parameters, it has another potential advantage: an array-oriented program in ELI is easier to be parallelized by a compiler for multi-core machines without user's involvement (see [6]). This could be quite attractive as we know that to parallelize a program is typically a painstaking and time consuming task.

2.2 One-liner Functions

2.2.1 number conversion and lexicographic ordering

Given a sequence x of hexadecimal digits, how do we convert x into its binary representation? Let us first introduce the primitive function ***encode*** `r<.c`; for scalar `c` and vector `r` (called the radix vector), it produces a result of shape `#r` which encodes `c` in the numerical system of `r`. For example,

```
      24 60 60<.3670
1 1 10
```

means that `3670` minutes is one day one hour and ten minutes; and

```
      2 2 2 2<.12
1 1 0 0
```

is the binary representation of `12`. The encode function extends to array `a` in such a way that the result is of rank `1+##a` and each column vector along the first axis is the result of applying `r` to that element of `a`. For example,

```
      2 2 2 2<.12 9 5
1 1 0
1 0 1
0 0 0
0 1 1
```

For a string of hexadecimal digits `c` we first convert it into a vector of numbers by querying the relative positions of the digits in the standard character list of hexadecimal digits (we must set `[]IO` to `0` first) using the *index of* function:

```
      []IO<-0
      '0123456789ABCDEF'!'C95'
12 9 5
```

We then apply encode to it, but the binary form of each digit is in a column. So we flip the resulting matrix and then ravel it. Hence, the function is the following

```
     {x2b:[]IO<-0; ,&.(4#2)<.'0123456789ABCDEF'!x}
x2b
      x2b '89ABCD'
1 0 0 0 1 0 0 1 1 0 1 0 1 0 1 1 1 1 0 0 1 1 0 1
```

We note that there is an inverse function to *encode*, i.e. the ***decode*** function `r>.c` (sect. 1.9 in Primer [4]). For example,

```
      24 60 60>.1 1 10
3670
      2 2 2 2>.1 1 0 0
12
      2>.1 1 0 0
12
```

We see that the difference is that if the left operand is a scalar then it will automatically expand to be a vector of the length of the right operand.

Let `TABLE` be a `8` by `n` array of names, i.e. each `TABLE[i;]` is a string of 8 characters of English alphabets and digits with trailing blanks. We want to sort `TABLE` into *lexicographical order*. There are two monadic *sort functions* ***grade up*** `<` and ***grade down*** `>` in ELI. For a numerical vector `v`, `<v` (resp. `>v`) gives the indices of elements of `v` in a non-decreasing (resp. non-increasing) order rearrangement. For example (we assume here `[]IO=0`):

```
      v<-10?.100   //pick 10 distinct numbers randomly from 0..99
      v
5 52 67 0 38 6 41 68 58 93
```

```
      <v
3 0 5 4 6 1 8 2 7 9
      v[<v]
0 5 6 38 41 52 58 67 68 93
```

The grade up and grade down functions in ELI also apply to character vectors with respect to lexicographical order. For example,

```
      c<-'awgnkn09'
      <c
6 7 0 2 4 3 5 1
      c[<c]
09agknnw
```

To solve our problem, we first note that the grade up function only applies to a vector argument. The following function extends `<` *along the last axis*, i.e. it gives indices of sorted elements in each row:

```
     {mgrdu: s#,.<"(w<-_1^.s<-#x)||,x}
```

What the function does first is to turn the input matrix into a list of rows using the ***partition*** function `||`(sect. 2.1 [4]). It then apply the *each* operator `"` to the grade up function `<` on each row and put the resulting list of rows back into a matrix using the ***raze*** function `,.` (sect. 2.1 [4]). To illustrate,

```
      m<-8 5#40?.100
      m
82 12  1 68 86
62 73 72 99 88
23 30 35 51 59
84 41 26 53 46
28 17 15 57 80
 3 49 95 74 55
89 21 71 13  9
27  0  2 70 93
      mgrdu m
2 1 3 0 4
0 2 1 4 3
0 1 2 3 4
2 1 4 3 0
2 1 0 3 4
0 1 4 3 2
4 3 1 2 0
1 2 0 3 4
```

However, to get the relative size (weight) of each row in `m` we need to apply `mgrdu` again:

```
      mgrdu mgrdu m
3 1 0 2 4
0 2 1 4 3
0 1 2 3 4
4 1 0 3 2
2 1 0 3 4
0 1 4 3 2
4 2 3 1 0
2 0 1 3 4
```

Now suppose that above is from a grade up of a flipped `8` by `n` name table, to put that in a lexicographical order

where the order of the first character dominates the second and so on the numeric weights of the names is calculated by the decode of the numbers representing the weight from ordering at each row. Hence, we have

```
       8>.mgrdu mgrdu m
4203553 2658906 6635784 1160915 9308292
```

Therefore, the solution to our problem is the following function:

```
       {lexord:[]IO<-0;x[<8>.mgrdu mgrdu &.x;]}
       table
abUni95w
jklmnboa
bUni95wj
klmnboab
Uni95wjk
      lexord table
Uni95wjk
abUni95w
bUni95wj
jklmnboa
klmnboab
```

2.2.2 a queuing network model

There is a PASCAL program in chapter 6, Performance Analysis, Desrochers [7], which implements the Boyse and Warn closed queuing network model. We shall only compute up to the average CPU queuing length *lq*. The reader who wants to go further will find that the whole 4-page program there can be put in 'one-line' ELI code once lq is calculated. By 'one-line' we mean that it may not be physically fit, or desirable, to be squeezed into one line but there is no transfer of control when we break it into several lines. The inputs to the model are

c	number of CPUs
k	multiprogramming level, c<k
e_o	mean service time for each I/O processor
e_s	mean service time fot each CPU

What to be computed are

p_o_0	probability of 0 jobs in the system
p_o_h	probability of h jobs in the system
ratio(j)	ratio of p_o_n/ p_o_0 for j:=1..k
lq	average CPU queue length

The PASCAL program segment to compute the *ratio(j)* is

```
if j<=c then
   ratio:=factorial(k)/(factorial(j)*factorial(k-j)*power(e_s/e_o,j)
else
   ratio:=factorial(j)/(factorial(c)*power(c,j-c))
              *factorial(k)/(factorial(j)*factorial(k-j)*power(e_s/e_o,j)
```

To put this in one-line ELI, we have to remove the use of if-then-else which would have been easy except for the fact that for `j>c` there is an extra factor multiplying into it. Let `jl<-!k` (i.e. `1...k`). The tentative code is

```
   ratio<-temp*((|.k)%(|.jl)*|.k-jl)*(e_s%e_0)*.jl
```

where `temp` is a vector of length `k` with first `c` elements `=1`, and elements from `c+1` to `k` equal to

```
factorial(j)/(factorial(c)*power(c,j-c))
```

Let `jtl<-c+!k-c`. Then `jtl` is a vector of length `k-c` ranges from `c+1` to `k`, and the tail section of `temp` above is

```
jtvl<-(|.jtl)%(|.c)*c*.jtl-c
```

Therefore, `temp` is

```
temp<-(c#1),jtvl
```

and `r=ratio` is

```
r<-((c#1),(|.c+jt)%(|.c)*c*.jt<-!k-c)*((|.k)%(|.jl)*|.k-jl)*(e_s%e_o)*.jl
```

p_o_0 is

$p\_o\_0 = 1/(1+sum)$

where *sum* is the summation of ratio over all j. In ELI, we have

```
p_o_0<-1%1++/r
```

Note that in ELI we use the ',' primitive to glue different formulae to replace a loop with *if-then-else* in PASCAL and use the reduction operator to avoid a loop in summation. According to [7] the probability of *h* jobs in system and the average CPU queue length is computed in PASCAL as follows

```
lq:=0
   for i:=1 to k do
       begin p_o_h:=ratio(j)*p_o_0;
            if i>c then lq:=lq+((i-c)*p_o_h)
       end;
```

So in ELI with `p_o_h` now stands for a vector from j=`1` to `k`,

```
p_o_h<-r*p_o_0
```

and `lq` is the summation of a vector `qv` of length (`k-c`) whose right factor is a portion of `p_o_h` from `i=c+1` to `k`:

```
qv<-(!k-c)*(c-k)!.p_o_h
```

Now remember that the *take* function `(-j)!.v` will take elements from rear of `v`. To put it all together, the function `BW_qmodel` below computes the average CPU queue length `z= lq` in the Boyes and Warn closed queuing network model where `o= e_o` , `s= e_s` and `p= p_o_h`:

```
@.z<-c BW_qmodel (k;o;s)
  z<-+/jt*(c-k)!.p<-r*1%1++/r<-((c#1),(|.c+jt)%(|.c)*c*.jt<-!k-c)*((|.k)%(|.jl)*jl)*(o%s)*.jl<-!k
@.
```

2.3 Control Structures

So far we have only given examples of straight-line codes. To write code involves alternatives or iterations classical APL [9] provides *branching* and ELI provides that as well; and that only appears in defined function definition. The simplest *branch statement* is

```
->0
```

that means function return. Another is

```
->msg
```

where `msg` is a character string and activation of this statement will output `msg` and then stop ELI execution. This statement is used when a program runs into exceptions. For example, if a piece of code runs into

```
->'name not found'
```

then ELI will emit the error message `'name not found'` and stop execution. A more general branch statement is of the form

```
->(boolexpr)/L
```

where `boolexpr` is an expression of boolean value and `L` is a label as statement in ELI can have a label like

```
L: expre
```

if `boolexpr` is `0` execution will continue to the next statement else the execution will branch to the statement with label `L` (see [4] for more forms of branching).

However, ELI provides *control structures* similar to those in C. In ELI a *simple statement* is either a line of code resulting in an expression as we described in **2.1** or a branch statement. A *statement* is either a statement or a group of (multi-line) statements bracketed by a pair of `{}`. Other than simple statement, ELI has the following statements

if-statement: **`if`** (*boolean expression*) *statement* [[**`else if`** (*bool expression*)]*"* **`else`** *statement*
case-statement: **`case`** (*case expression*) {*case-lists* [**`else`** *statement*]}
where *case-list : v1*[**,***v2..vn*]**:** *statement*
for-statement: **`for`** **(***idxv***:***for-forment***)** *statement*
where *for-forment*: *strv***;***endv*[**;***step*] or *idxlist*
while-statement: **`while`** **(***boolean expression***)** *statement*

In a case statement, *case expression* must result in a *discrete scalar type*, i.e. an integer, a character or a symbol.

We shall see their usage through various examples later; but we first give a recoding of the 31-line APL function `OPERATION` in Digital System Implementation [1] p.138 which specifies the set of System/360 operations. `inst` is a boolean vector representing an instruction where bits `4-7` is the portion for op-code with comments after line(s) being the op-code name; `od1` is the first operand, `od2` the second operand and `rl1` the result, all in bits. `TWOC` is the 2-complement interpretation function and `ARCHDIV` is the divider architecture. Note the result of multiplication is a 64-bits vector putting into two pieces of 32-bits code, and in division a double precision dividend `dd` is used which is

comprised of two operands `od2` and `od3`. An interested reader can compare the following code with that in [1] and see clearly that the case statement in ELI makes the code more intuitive and close to the original intention.

```
@.OPERATION
  case (2>.inst[4 5 6 7]) {
  //data handling
    0: rl1<-(32#2)<.|TWCO od2              //POS
    1: rl1<-(32#2)<.-|TWCO od2             //NEG
    3: rl1<-(32#2)<.-TWCO od2              //COMP
    2,8: rl1<-od2                          //LOAD
  //logic
    4: rl1<-od1^od2                        //AND
    6: rl1<-od1&od2                        //OR
    7: rl1<-od1~=od2                      //XOR
  //arithmetic
    10,14:rl1<-(32#2)<.(2>.od1)+2>.od2     //ADD
    5,9,11,15:rl1<-(32#2)<.(2>.od1)-2>.od2 //SUB
        12:{pd<-(64#2)<.(TWCO od1)*TWCO od2 //MPY
            rl1<-32^.pd
            rl2<-32!.pd}
        13:{dr<-od1                         //DIV
            dd<-od2,od3
            ARCHDIV
            rl1<-qt
            rl2<-rm}
  }
@.
```

2.4 Recursion

2.4.1 sorting

Recursion is a powerful programming technique to solve some seemingly complicated problems in a systematic way. It is the main programming paradigm in LISP. In APL/ELI, we usually try to find non-recursive solutions in terms of array operations first before we try recursion. Nevertheless, recursion is an integral part of APL/ELI. In general, *recursion* applies to data which are irregular, or not array like. But recursion can still be applied to vectors and arrays to yield some elegant solutions. Let us start with a sorting example.

We already know from sect. 1.5 that there is a lexicographical ordering among characters. We shall define an ordering *gre_eq* (>=) between two character strings *s1* and *s2* as follows where *si*[1] is the first character of *si*:

```
  if s1[1]>s2[1] then s1>= s2 is 1
  if s1[1]<s2[1] then s1>= s2 is 0
  if s1[1]=s2[1] then compare 1!.s1 with 1!s2
  empty string < non-empty string
```

Hence, our comparison function is as follows:

```
@.z<-le gre_eq ri
  if (0=^le)
    if (0=^ri) z<-1 else z<-0
  else if (0=^ri) z<-1
  else if ((l1<-1^.le)>r1<-1^.ri) z<-1
  else if (l1=r1) z<-(1!.le) gre_eq 1!.ri
  else z<-0
@.
      'bab' gre_eq 'b'
```

```
1
      'bab' gre_eq 'cb'
0
      'b' gre_eq 'b0'
0
```

The problem we want to solve is how to sort a vector of symbols such as

```
      v<-`abc `wx `a0 `c123 `cd
```

where the order of each pair of symbols is determined by the order of their character string representations as we just defined above. Hence, the steps are

1. turn `+.v` into a list `sl` of character strings representing the symbols
2. apply the *quicksort algorithm* to the list `sl`
3. put the sorted list of strings back into a vector of symbols

To do 1, we observe that

```
      +.v
abc wx a0 c123 cd
```

We see that it is a vector consists of characters denoting symbols in `v` separated by a blank but with ‘`’ taking out. We append a blank in front of it and then get a boolean vector indicating blank or non-blank. We then use the monadic function *partition count* `||` to get a count of each string starting with a blank; from there we apply the *count* function to the slightly modified `v` in dyadic *partition* `||` to get a list of strings each with a blank affixed in front of a character string representing a symbol in `v` and we then apply `1!.″` to drop that blank.

```
      ' '=' ',+.v
1 0 0 0 1 0 0 1 0 0 1 0 0 0 0 1 0 0
      ||' '=' ',+.v
4 3 3 5 3
      (||' '=v1)||v1<-' ',+.v
< abc
< wx
< a0
< c123
< cd
      1!."(||' '=v1)||v1<-' ',+.v
<abc
<wx
<a0
<c123
<cd
```

The ***quicksort*** algorithm is invented by Tony Hoare. The basic idea is very simple. It takes one element x from a list L and divide L into two piles: *pile1* consists of elements less than x and *pile2* consists of those greater than or equal to x; it then applies the same procedure *recursively* to *pile1* and *pile2*. Hence, our code is (where x is just the first element of L)

```
@.z<-sortsyms v;v1
  z<-!.,.’`’,” quicksort 1!."(||' '=v1)||v1<-' ',+.v
@.

@.z<-quicksort ls;b
```

```
  if (1>=#ls) {z<-ls
    ->0}
  z<-(quicksort (~b)/ls),quicksort (b<-ls gre_eq" 1^.ls)/ls
@.
```

`b` is the result of `gre_eq` comparison of each element in `ls` with `ls[1]` and `b/ls` is the compression of `ls` by `b`. We note that for a list `z` of character strings, `'`',"z` affix a `'`'to each element in `z`, and the *raze* function `z,.` just ravels a list of homogeneous data back into a vector. For example,

```
      ,.(2 5 7;8 10;9; 22 100 _3 1)
2 5 7 8 10 9 22 100 _3 1
```

And we already mentioned in sect. **1.3** that the execute function `!.c` will turn certain character vector `c` into a vector of symbols.

However, the code thus presented has a fatal error which is not easy to find out by testing which by nature can only run a finite number of times. To *prove* the ***correctness*** of a program *P*, we must first prove the *P* will end, i.e. it will terminate. There is no question that a straight line ELI code (segment) with only primitive functions and derived functions will terminate. Hence, `sortsyms` will terminate if and only if `quicksort` will. For a recursive function `rf` to terminate the recursive calls inside `rf` must be applied to data of *smaller* pieces, for otherwise the chain of recursive calls will be endless. Here we find the problem: it is entirely possible that `ls[1]` is the '*smallest*' of `ls` thus `b` is all `1` and `b/ls` is `ls` again. Once we realize the problem, to fix the bug is quite simple:

```
@.z<-quicksort ls;b;w
  if (1>=#ls) {z<-ls;->0}
  z<-(quicksort (~b)/w),l1,quicksort (b<-w greeq" l1<-1^.ls)/w<-1!.ls
@.
```

What we did is taking the first element out of the dividing process and put it in the middle of two piles, one with smaller elements and one with larger or equal elements. Now the arguments to the recursive calls must have length less than that of `ls` because `w` itself is of length one less than that of `ls`.

We make a comment on the effort to prove program correctness here. As Dijkstra pointed out that no amount of testing can vouch for the correctness of a program. Hence, the goal of proving program correctness is certainly noble and recently there are substantial successes in its application to the area of OS and hardware implementations. Nevertheless, for most good size programs, especially C programs with pointers, this remains a formidable task and only of academic interest. What APL/ELI brings to the table is that by subsume many if-then-else and loops into high-level primitives and put substantial portion of code into one-line segments it reduces the logical complexity of a program considerably as we have seen in the formulation of the function `sortsyms` above. Therefore, succinctness of ELI code is not just a matter of esthetic taste but brings clarity which reduces error and helps program reasoning.

There is another slightly different way to sort a list of symbols without using list. We turn the vector of symbols into a matrix of rows of character strings of equal length and apply the quicksort algorithm by taking the first row and compare it with the rows in the rest of the matrix. Once the character matrix is sorted, we turn it back into a vector of symbols by first affixing "`" in front of it and then applying the execute function `!.`.

```
      ((^v),1)#v
`abc
`wx
`a0
`c123
`cd
```

```
      +.((^v),1)#v
abc
wx
a0
c123
cd

@.z<-quicksort2 ls;b;w;i;r;s;l1
  if (1>=r<-^.s<-#ls) {z<-ls;->0}
  w<-1 0!.ls; l1<-ls[[]IO;]
  b<-(r-1)#0
  for (i:1;r-1) b[i]<-w[i;] gre_eq l1
  z<-(quicksort2 (~b)/.w),.l1,.quicksort2 b/.w
@.

@.z<-sortsyms2 v;v1
  z<-!.,'`',quicksort2 m<-+.((^v),1)#v
@.
     sortsyms2 v
`a0 `abc `c123 `cd `wx
```

2.4.2 tower of Honoi

The Tower of Honai is a game with three poles A, B, C and a set of *n* disks fit onto a pole, say A, by means of a hole cut through the center of each disc. The discs are of increasing size from top to bottom. The game is to move all discs from A to C one disc at a time, but a larger disc should never put on top of a smaller one. B can be used as an intermediate resting place for transient discs. The problem looks complicated until we assume that that a sub-problem has already been solved. Suppose we can find a way to move the first *n-1* discs from pole A to pole B. We only need to move the *n*-th disc from A to C, and then move the *n-1* discs from B to C. Let

```
k discmove poles
```

be a function call to move `k` discs from `poles[1]` to `poles[3]` using `poles[2]` as an intermediate place (`poles` is represented by three characters A,B,C). The program is

```
@.n discmove poles
  if (n=1) {
    []<-'move the first disc from '
    []<-poles[1]
    []<-' to '
    []<-poles[3]
       ->0}
  (n-1) discmove poles[1 3 2]
  []<-'move '
  []<-n
  []<-'-th disc from '
  []<-poles[1]
  []<-' to '
  []<-poles[3]
  (n-1) discmove poles[2 1 3]
@.
```

We note here both `[]<-expr` (called *quad output*) and `()<-expre` (called *bare output*) are to output the right side `expr` in ELI. The difference is that for the bare output the next output will follow where it left off, i.e. no line-end

character will be at the end of this output. The termination condition for the `discmove` function is clear as the recursive call is with left argument `n-1`. We take a sample run of three discs:

```
       3 discmove 'ABC'
move the first disc from A to C
move 2-th disc from A to B
move the first disc from C to B
move 3-th disc from A to C
move the first disc from B to A
move 2-th disc from B to C
move the first disc from A to C
```

2.4.3 determinant

For a square matrix `m`, we would like to find its *determinant* `det m` recursively. First, if `m` is of rank `1`, i.e. `m` is of the form `[a]` then is `a`. Next if `m` is of rank `2`, i.e. `m` is of the form

```
|a b|
|c d|
```

then `det m` is `ad-bc`. For a general `m`, if we can rearrange `m` into a form with its first row all `0` except the first a_{11}, and let us denote the sub-matrix of `m` formed by rows and columns from second on m_1. Then `det m` is a_{11}`*det` m_1. Of course, the rearrangement needs to leave the determinant invariant. The code is the following:

```
@.z<-det m;n;p;i         //det of a matrix nxn to nxn.
  i<-[]IO
  case (n<-1^.#m) {
    1: z<-m[i;i]
    2: z<-(m[i;i]*m[i+1;i+1])-m[i;i+1]*m[i+1;i]
    3: z<-+/(*/.0 1 2$m)-*/.2 1 0$m
    else {
      if (^/m[i;]=z<-0) ->0
      if (0=m[i;i]) {m[;p,i]<-m[;i,p<-(0=m[i;])!0]
        m<--m
      }
      m<-m-(m[;i]*%z<-m[i;i]).:*0,m[i;1!.!n]
      z<-z*det 0 1!.1 0!.m
    }
  }
@.
```

Here for the case rank=3, we employed a direct formula to calculate the result. The reader can check a linear algebra book to verify that the code indeed implement the formula where for example,

```
      m
1 2 3
4 5 6
7 8 9
      2 1 0$m
3 1 2
5 6 4
7 8 9
      0 1 2$m
1 2 3
5 6 4
```

```
9 7 8
```

the rotations rotate each row of `m` a different amount and `*/.` is column-wise product while `+/` is the sum. In the `else` part covering the general case, we first check whether the first column is all `0` and in that case the result is `0`. In case the left upper corner element is `0`, `p<-(0=m[i;])!0` find the first non-zero element in the first row `m[i;]` and then exchange the corresponding columns by `m[;p,i]<-m[;i,p]`. Since this exchange will change the sign of `det m` we do `m<--m`. The next line reduce all other elements in the first row to 0, and the last line makes the recursive call to the sub-matrix of `m` formed by dropping the first row and first column of `m`. For a test run,

```
      m
 4 _7 27  7 22
28 40  9  3 23
_6 35 12 29 14
 2  8 _1 15 36
16  6 _5 38 10
      det m
_5965161
```

2.5 Script Files

A user can prepare his/her ELI code in an ordinary text file of extension `txt` or `esf`. A script file can contain function definitions in full format or in short-form. For a function `sf` defined in short-form any function `f` called inside `sf` must be defined before the definition of `sf`. A script file can also contain definitions of variables and executable statements. To input a variable in a file, one write a head line first

```
&vnam type rank shape
```

where `vnam` is the name of the variable, `type` is a character indicating the type of the variable (‘`B`’ for boolean, ‘`I`’ for integer , ‘`E`’for floating-point, ‘`C`’ for character, ‘`S`’ for symbol etc. see [4] for others), `rank` and `shape` are the rank and shape of `vnam`. Starting from the next line is the *raveled* value of `vnam` where elements are separated by a blank space except in the case of character variable for which blanks are significant, but not required in case of symbol type. A variable input is ended by a line with a single ‘`&`’. For example for the variable

```
      m3
1 2 3
4 5 6
7 8 9
```

the script input is

```
&m3 I 2 3 3
1 2 3 4 5 6 7 8 9
&
```

If we have large input from a source other than hand prepared text, we only need to insert head line and end line to make it into a legitimate input to ELI.

ELI variables and functions can also output to a text file from an ELI session or a loaded ELI workspace by the command

```
    )out file1 fn1 fn2 a b
```

This would result in a script file `file1.esf` containing functions `fn1 fn2` and variables `a b`. If there is no name following the file name `file1` then everything in the session or workspace will be output to the file `file1`.

ELI system comes with a script `standard.esf` which contains many useful pre-defined functions. Hence,

```
)fcopy standard
```

will copy in the script file, then a group of frequently used utility functions are available. See **section** 4.4 in [4] for examples.

3. Array Implementation of Data Structures

3.1 Emulation of PASCAL Data Structures in ELI

3.1.1 a small database for a company

Suppose we want to build a database of personnel information for a company C. The information for each employee of this company is represented as a record in the PASCAL programming language as follows:

```
type employee= record
                 name: array [0..7] of char;
                 phone: array [0..11] of char;
                 division: array [0..7] of char;
                 manager: ↑ employee;
                 subordinates:↑ employeelist;
                 salary: integer;
                 sex: (male, female);
                 birthdate: array [0..2] of integer;
                 status: (fulltime, parttime, fired, retired);
                 yearhired: integer;
                 yearterminated: integer;
               end;

type employeelist= record
                     first:↑ employee;
                     next: ↑ employeelist;
                   end;
```

where indicates that `manager` and `first` are fields of **pointer** type points to record of type `employee`. The type is just PASCAL's way of implementing a linked list (see Wirth [14] sec.4.2), and the field

```
subordinates:↑ employeelist;
```

represents the list of employees directly reporting this employee in case he is a manager. We encounter three immediate problems when try to implement this database in a way similar to the PASCAL case:

1. there is no record structure or ***struct*** in ELI,
2. there is no pointer type in ELI,
3. ELI does not have the build-in procedure ***new***(p) of PASCAL to dynamically create an item of the type the pointer p points to.

Before we set to solve these problems, let us reflect on what we want: a data structure where an employee's record can be created for new hires, status change due to promotion, demotion, firing or retiring can be easily updated, and the hierarchy of the company should be evident in the database. Moreover, we would like to have various queries about the company and its divisions answered quickly.

First we observe that the database can be implemented in PASCAL as

```
array [0..totalnumber] of employee;
```

where `totalnumber` is the total number of employees in the company. But it has one problem in PASCAL: to add a new employee we have to declare another array of length `totalnumber+1`. In ELI, there is no array declaration and

an array `A` can be appended by a new item simply by:

```
A<-A,newitem
```

Since a record in PASCAL is in essence a collection of fields, we can use a collection of arrays in ELI to emulate an array of records in PASCAL (in this chapter, we use `[]IO=0`):

```
ename<-(totaln,8)#' '
ephone<-(totaln,12)#' '
ediv<-(totaln,8)#' '
emangr<-totaln#_1
esubos<-totaln#_1
esubon<-totaln#0
esalary<-totaln#0
esex<-totaln#' '
ebdate<-(totaln,3)#0
estatus<-totaln#' '
ehire<-totaln#0
eterm<-totaln#0
```

where `totaln` is the total number of employees. Then each record in the original PASCAL implementation is represented by a collection of values of arrays at a specific index, say `i`. For example, the array `ename` is a `totaln` by `8` matrix of characters, and

```
     ename[i;]
tramble
```

represents the (truncated or padded with blanks to 8 characters) name of an employee in the i-th slot of the database; and

```
     ephone[i;]
914-456-7688
```

is the phone number of that employee. The array

```
emangr
```

which corresponds to a field of pointer type in PASCAL is initialized to `_1` which stands for the ***nil*** value in PASCAL; and

```
emangr[i;]
```

is an index `j` between `0` and `totaln` which represents the manager of `tramble`(i.e. if `tramble` has a manager) named

```
ename[j;]
```

If `tramble` is the CEO of the company, then he has no manager and `j=_1`. Note that there are two arrays `esubos`, `esubon` correspond to the field

```
subordinates:↑ employeelist;
```

in PASCAL. This is because we decide to use a numeric array

```
employeelist
```

of indexes to represent the corresponding linked list in PASCAL such that `esubos[i]` points to a starting location in `employeelist` of a list of indexes representing employees who directly report to `trimble`, and `esubon[i]` is the length of that list. Hence, the list of indexes of employees reporting to a manager if index `i` is

```
employeelist[esubos[i]+!esubon[i]]
```

and the name of these employees can be printed by

```
ename[employeelist[esubos[i]+!esubon[i]];]
```

While ELI has *symbol* type, here we simply use two characters 'm' and 'f' to represent (male, female) and 'f', 'p', 'x', 'r' to represent the values of the corresponding field of `status`. To add a new employee

```
name: Johnson
phone: 914-456-8366
division: Reseach
..
status: fulltime
yearhired: 2003
```

we do

```
ename<-ename,.8^.'Johnson'
ephone<-ephone,.'914-456-8366'
ediv<-ediv,.'Research'
..
estatus<-estatus,.'f'
eyhire<-eyhire,.2003
```

and the old `totaln` becomes the employee identification number in this database while

```
totaln<-totaln+1
```

At this point we want to introduce the concept of ***cost*** of a computation informally. Each ELI primitive operation is implemented by a group of machine operations (codes), and some are more costly than others in terms of execution time or storage requirement. For example, the operation

```
a<-a,newpart
```

is far more costly than the operation

```
i<-i+1
```

regardless of whether `newpart` is a scalar (i.e. a single item) or a large array. Not getting into the fact that ELI is interpreted and there is an interpreter-overhead associated with the execution of any ELI expression, this can be intuitively explained as follows. While the second expression only involves adding `i` by `1`, which can basically be accomplished by one machine instruction, the first expression involves the creation of a new array in the computer memory, copying the values of the old array to the new array area and then copying the value of `newpart` next to it. The ability to device a more efficient program with the same functionality is what separates an experienced ELI/ APL programmer from a beginner. All one-line examples in the last chapter can be implemented with loops in the C/ PASCAL fashion. The reason an experienced APL programmer is likely to choose an array-oriented solution is

mostly due to the fact that a one-liner executes far faster than a loopy solution under an interpreter system, especially when computer memory is tiny. And this is how historically an array-oriented programming style developed in the APL community. However, an unforeseen advantage is that an array-oriented program is more amenable to automatic parallelization under an advanced compiler (see Ching and Zheng [6]).

Back to the programming task at hand, we realize that by adding a new item to each component array in our database when adding a new employee is a bad design if the company is likely to add new employees fairly frequently. An alternative design would be to allocate a collection of arrays to a size a bit larger than the expected size of the company and keep tracking what it the next employee identification number which is the same as the total number of employees ever recorded in the database since we choose to use `[]IO=0`. Hence, we have

```
@.initialize
  nxtempn<-nxtsen<-0
  maxnem<-1000+incnemp<-1000
  ephone<-(maxnem,12)#' '
  ediv<-(maxnem,8)#' '
  emangr<-maxnem#_1
  esubos<-maxnem#_1
  esubon<-maxnem#0
  esex<-maxnem#' '
  ebdate<-(maxnem,3)#0
  estatus<-maxnem#' '
  eyhire<-maxnem#0
  eyterm<-maxnem#0
  employeelist<-maxnem#_1
@.
```

to initialize the database. To insert a new employee record, we first check whether our table (i.e. the collection of arrays is large enough to contain the new record. In case space is already exhausted then we increase the size of the table by a fixed, but large chuck. The function is

```
@.z<-inc
  if (maxnem>=1+z<-nxtempn) ->0
  else { //increase the table by a chunk of size incnem
    ename<-ename,.(incnem,8)#' '
    ephone<-ephone,.(incnem,12)#' '
    ediv<-ediv,.(incnem,8)#' '
    emangr<-emangr,incnem#_1
    esubos<-esubos,incnem#_1
    esubon<-esubon,incnem#0
    esex<-esex,incnem#' '
    ebdate<-ebdate,.(incnem,3)#0
    estatus<-estatus,incnem#' '
    eyhire<-eyhire,incnem#0
    eyterm<-eyterm,incnem#0}
@.
```

Under this new organization, the previous segment of code to enter information for new employee `Johnson` becomes

```
  ename[i<-inc;]<-8^.'Johnson'
  ephone[i;]<-'914-456-8366'
  ediv[i;]<-'Research'
  ..
  estatus[i]<-'f'
  eyhire[i]<-2003
```

3.1.2 implementation of linked lists in ELI

We want to study our implementation of a linked list using two component fields `esubos`, `esubon` and a storage array `employeelst`. For the sake of simplicity, let us assume for the moment that all employees have unique names which are at most eight characters long. Suppose the new employee `Johnson` is reporting to a manager named `Brook` and one of his subordinate named `Peters` has just left. Recall from section 1.8 (`T` tnow is `ename[!nxtempn;]`, the matrix containing names of all employees ever employed by company C) that the expression assigned to `z` in the following function

```
@.z<-find nam
  if (nxtempn<=z<-(ename[!nxtempn;]^:=8^.nam)!1)
    []<-'No employee named ',nam,' in this company'
@.
```

returns the identification number of an employee with name `nam`, and when `z=nxtempn` means no match of the name `nam` is found. Hence, first we do

```
emangr[i]<-j<-find 'Brook'
```

To list all employees directly report to a manager named `mnam`, we have

```
@.manager_g mnam
  []<-enames[employeelst(esj<-esubos[j])+!esubon[j]]
@.
```

But we also need to update the field of subordinates for `Brook`. A simplistic approach would be to find where `Peters`'s identification number is located in `employeelst` and replace it by `Johnson`'s id-number:

```
employeelst[employeelst!find 'Peters']<-i
```

However, as we will explain later, the id-number of `Peters` may appear in multiple instances in `employeelst`. Therefore, a correct way to do updating is searching `Peters`'s id-number thru the sublist of subordinates of manager with id-number `j`:

```
slist<-employeelst[(esj<-esubos[j])+!esubon[j]]
```

and replace accordingly:

```
employeelst[esj+slist!find 'Peters']<-i
```

Now in case `Peters` left company, or started reporting to another manager, without any replacement, we must adjust

```
esubon[j]<-newsen<-esubon[j]-1
```

More than that we must adjust the list `slist` above to eliminate the id-number of `Peters` from the list everywhere it appears, and reassign to the storage array `jemployeelst` as follows:

```
employeelst[esj+!newsen]<-(slist~=find 'Peters')/slist
```

We note that the last id-number in `slist` is still in `employeelst` but becomes inaccessible because `esubon[j]` is 1 less than previously. This is why an id-number may appear more than once in `employeelst` due to such updating.

Next, suppose `Johnson` is simply a new hire, not a replacement. So his manager just has one more employee reporting to him/her:

```
esubon[j]<-newsen<-esubon[j]+1
```

But we cannot just append the `slist` in `employeelst` by the id-number `i` because that slot may well be occupied by the id-number of an employee who reports to another manager. We need a fresh segment in `employeelst` to store the newly expanded list `slist` representing employees reporting to `Brook`. And to do that we need another function similar to `inc`:

```
@.z<-inx n
  if (maxnem>=nxtsen<-n+z<-nxtsen) ->0
  else  // increase the list by a chuck of size incnemp
    employeelst<- employeelst,incnemp#_1
@.
```

This function takes an argument `n` because we anticipate situations where a manager may increase the number of subordinates by more than one. `inx` returns the first index of a size `n` segment in `employeelst` which is not occupied as we use `nxtsen` to keep track of where new segment can start. Anyway, back to our case of adding `Johnson`, we have

```
employeelst[(inx 1)+!newsen]<-slist,i
```

We realize that this use of pointer-vector and length-vector combination with one store-away array for implementing a group of lists has serious draw-back: any time a particular list grows, a fresh storage-area will be needed and a copy operation is required. An alternative approach is to use a `imaxsen` by `2` matrix

```
employeelink<-(maxsen,2)#_1
```

so that for a manager with id-number `j`

```
employeelink[jsx<-esubos[j];0]
```

points to (i.e. it is the id-number of) the first employee reporting to that manager and if

```
employeelink[jsx;1]
```

is not `_1` then it points to a row in `employeelink` where the next employee (i.e. his id-number) reporting to the same manager is located:

```
employeelink[employeelink[jsx;1];0]
```

is the id-number of that next employee. This chain of ***linked list*** of employees reporting to manager `Brook` ends at a slot `sx` with

```
employeelink[sx;1]=_1
```

We still use `nxtsen` to keep track the next unused slot in `employeelink`, but we no longer need the component field `esubon` to keep track of the length of a list. This implementation of a ***linked list*** is structurally similar to the PASCAL's type definition of **type** `employeelist`.

The function `inx` is now changed to

```
@.z<-inx1
  if (maxnem>=nxtsen<-n+z<-nxtsen) ->0
  else  // increase the list by a chuck of size incnemp
    employeelink<- employeelink,.(incnem,2)#_1
@.
```

To list all employees directly report to a manager named `mnam`, we have

```
@.manager_g mnam
  if (_1=sx<-esubos[find mnan]) {[]<-'No one report to ',mnam;->0}
  []<-'The following people report to ',mnam
  while (_1~=sx) {
    []<-ename[employeelink[sx;0]
    sx<- employeelink[sx;1]}
@.
```

To add an employee `enam` to the list of employees reporting to a manager named `mnam` we do

```
@.enam add_togp mnam
  i<-find enam
  if (_1=sx<-esubos[j<-find mnan])
    employeelink[esubos[j]<-lx<-inx1;0]<-i
  else {
    while (_1~=sx) sx<-employeelink[sx;1]
      employeelink[employeelink[sx;1]<-inx1;0]
  }
@.
```

To delete an employee named `enam` from the list of people reporting to a manager named `mnam`, we do

```
@.enam drop_fmgp mnam
  if (_1=sx<-esubos[find mnan]) {[]<-'No one report to ',mnam;->0}
  else if (_1=employeelink[sx;1]){
    esubos[j]<-_1 // only one employee reporting to mnam
    ->0}
  else {
    i<-find enam
    sx0<-sx
    while (_1~=sx) {
      if (found<-i=employeelink[sx;0]) break
      sx<- employeelink[sx;1]
    }
    if (~found) []<-'No one named ',enam,' reported to ',mnam,'.'
    else if (sx=sx0) esubos[j]<- employeelink[sx;1]
    else employeelink[sx0;1]<- employeelink[sx;1]
  }
@.
```

So we see that the program to list all employees reporting to a manager or to eliminate an employee from such a list is more involved now, but we avoid copying a whole list of reporting employees when adding a new employee to

the corresponding list. Another possible alternative is to introduce a new component field `esubom` to reserve a segment of length

```
esubom[find newmangrnm]
```

in `employeelst` when someone named `newmangrnm` is made a manager, and `inX` is called only when an addition to a group of new subordinates will make the number of employees reporting to that manager larger than the prescribed segment length. We will leave the detail of implementation to readers as an exercise.

3.1.3 implementation of queries to a database

The reason people want to implement databases about some organizations or certain areas of inquiry is that we can efficiently find information related to that organization or area by making queries to the database of concern. There are generally three kinds of queries: ***individual***, ***global*** and ***departmental***. An *individual* query is about one person in the organization. For example, to find the phone number of an employee named `nam`

```
@.phone nam;i
  if (nxtempn<=i<-find nam)->'No employee named',nam
  else {
    ()<-nam,': '
    []<-ephone[i]
  }
@.

phone 'Johnson'
Johnson: 914-456-8366
```

Global queries are about the whole organization. For example, to find the number of full and part time employees, we have

```
currentemployees<-+/(flst='p')&'f'=flst<-status[!nxtempn]
```

And the total amount of salaries which the company pays is

```
totalsa<-+/((flst='p')&'f'=flst<-status[!nxtempn]/nxtempn^.esalary
```

Note that there only `nxtempn` entries in the database which have been ever used, but the list also contains the last salaries of former employees, i.e. those retired or fired. So we first have to set up a boolean mask to select entries of full-time and part-time employees using the field `status` and then add them up.

Not all global queries result in a single number. For example, the following function will list all employees who are paid more than twice of the average salary together with their salaries.

```
@.twicepay
  totalsa<-+/(mask<-(flst='p')&'f'=flst<-status[alst<-!nxtempn])/cursal<-nxtempn^.esalary
  lst<-(esalary[curlst]>2*totalsa%+/mask)/curlst<-mask/alst
  []<-ename[lst;],+.((#lst),1)#esalary[lst]
@.
```

`curlst` is the list of id-numbers of current employees, and `+/mask` is the total number of current employees. Hence, `totalsa%+/mask` is the average salary, and `lst` is the list of id-numbers of employees whose salary is more than twice the average. Recall from Chapter 1.6 that the *format* function `+.` turns numeric data into their character

representations. But applying `+.` to `esalary[lst]` directly will result in a vector of characters while we want it to be a matrix of characters (digits) such that each row is the salary of the employee with the same id-number. Therefore, we apply a reshape to turn it into a `lst` by `1` matrix before applying `+.` to it. A simple illustration is the following:

```
      a<-132 756 459 533 219 48 679 680 935 384
      +.a
132 756 459 533 219 48 679 680 935 384
      a1<-10 1#a
      a1
132
756
459
533
219
 48
679
680
935
384
      a1c<-+.a1
      a1c
132
756
459
533
219
 48
679
680
935
384
```

Let us write a related function: to list all employees (and their salaries) whose salaries are in the top 10% of all employees. First we note that we cannot simply look for a number and select ones whose salary is over that threshold; for this will not necessarily give us the top 10% of the employees. For our task, we literally have to sort people in terms of their salaries and then take the top 10% to be listed:

```
@.top10pc
  salist<-(mask<-(flst='p')&'f'=flst<-status[alst<-!nxtempn])/cursal<-nxtempn^.esalary
  lst<-(_.0.1*+/mask)^.(curlst<-mask/!alst)[<salist]
  []<-ename[lst;],+.((#lst),1)#esalary[lst]
@.
```

We recall that `curlst` is the list of id-numbers of all currently active employees and `<salist` is the sorted list of indexes from `0` to `(+/mask)-1` so that `salist[<salist]` is a list of salaries in descending order. Hence, `curlst[<salist]` is the list of id-numbers whose corresponding salaries are in descending order. Therefore, `lst` is the top 10% list we want. We note a slight blemish here: there could be people with the same salary while one is listed while the other is not. This is because we cut-off at the 10% boundary and a group of people with the same amount of salary may have their id-numbers listed across that boundary.

A *departmental* query is about a division of a company. For example, the following function with parameter `dnam`, the department name, lists all employees in that department with their names and phone numbers in alphabetical order:

```
@.deptphone dnam
  dlst<-(ediv[all]^:=8^.dnam)/all<-(estatus[all]?'fp')/all<-!nxtempn
  sdlst<-dlst[<27<.' ABCDEFGHIJKLMNOPQRSTUVWXYZ'!enames[dlst;]]
  []<-ename[sdlst;],ephone[sdlst;]
@.
```

The first line gets all id-numbers of employees belong to the department named `dnam`. The second line sorts that list into another list alphabetically as we explained in Chapter 2.4.3. We also remark that `[]<-` in the last line as well as in other examples is not really necessary. This is because the ELI interpreter will always display an expression if it is not assigned to a variable. We used `[]<-` merely to emphasize that an output is being carried out here.

Another example is that of calculating the ratio of managers to non-managers in a particular department named `dnam`. The following function gives the percentage of managerial employees in a department.

```
@.z<-deptmratio dnam
   dlst<-ediv[all]^:=8^.dnam)/all<-(estatus[all]?'fp')/all<-!nxtempn
   z<-(+/_1~=esubos[dlst])%#dlst
@.
```

Note that for an id-number `x`, `_1~=esubos[x]` indicates that `x` is an id-number of a manager. We see that an departmental query is really no different from a global query except the extra step to get the list `dlst` of id-numbers in a given department.

3.2 Binary Trees

3.2.1 tree representation

Trees, especially *binary trees*, have been used extensively in programming to organize data into a structure on which the operations of ***search***, ***deletion*** and ***insertion*** can be implemented efficiently. One also would like to visit every node of a tree in certain order. Trees are used in programming language implementation (parse trees), artificial intelligence (game trees), data retrieval (binary search trees) and many other areas of application. A ***binary tree*** is a tree whose node is either a leave node, i.e. a node without children, or has at most two *child-nodes*, a *left-son* and/or a *right son-node*. Each node of a tree typically also contains a *key* which may represent the data item stored at that node. The key can be a character string, a symbol, an integer or a bit-string of fixed length. Keys are ordered and can be compared. Typically, the left sub-tree contains keys less than or equal to keys of the nodes on the right sub-tree. The result of comparison of two keys, one given and one from a node encountered during a tree visit guides an insertion or a search to go to left subtree, quit or go to right subtree. In PASCAL terminology, we can declare a simple binary tree by

type tree = **record**
key: **array**[0..7] of **char**;
lson, rson: ↑tree;
end;

Following the approach of implementing such record structure in ELI, the function to initialize a tree structure is

```
@.inittree
  nxtndx<-0 maxndx<-1000+incndx<-1000
  tkey<-(maxndx,8)#' '
  tlson<-trson<-maxndx#_1
  alph<-' ABCDEFGHIJKLMNOPQRSTUVWXYZ'
@.
```

A tree is represented by its root-node's index. Initially, we have

```
root<-_1
```

And the function corresponds to **new**(tree) in PASCAL to create a new tree node is

```
@.z<-inct
  if (maxndx>=nxtndx<-1+z<-nxtndx) ->0
  else {// increase the table by a chuck of size inctree
    tkey<-tkey,.(inctree,8)#' '
    tlson<-tlson,inctree#_1
    trson<-trson,inctree#_1
  }
@.
```

Suppose the keys are to be ordered *lexicographically* and that characters appearing in keys are all in upper case English alphabets. Then as explained in section 2.2.4 and the monadic *signum* function `*` mentioned in section 1.4 the following function

```
@.z<-lkey compare0 rkey
  z<-*(27<.alph!lkey)-27<.alph!rkey
@.
```

compares two keys `lkey`, `rkey` in lexicographical order and returns a `1` if `lkey>rkey` in that order, `0` if equal and `_1` otherwise.

At this point we would like to discuss the concept of ***prototyping*** in a very high level programming language and low-level programming. As readers who have experience of programming in other programming languages such as FORTRAN, PASCAL or C/C++ can see that once one is fluent in APL/ELI, in general it is far faster to get a piece of program coded in APL/ELI and get it running than in a sequential language where the natural unit of operation is one cell and operations on an array are through loops over its elements. Therefore, historically APL was often used as a *prototyping language*, i.e. used to build prototypes but not the final production code. By using a language like APL to implement a prototype first has the advantage of checking out the basic algorithms and data structures in a program design quickly. This is particularly attractive if the eventual implementation language is decided ahead of time to be the assembly language of a particular machine or C. This is so because the APL interpreter provides a high level and convenient debugging environment while most assembly language or C debuggers are of low-level vista. Now most C/C++ comes with excellent IDE (*interactive development environment*). Nevertheless, debugging in APL/ELI is still easier than debugging a corresponding program in C because it lets programmers to concentrate on higher level abstractions in terms of primitives it provided (of course, there are many programming tasks which require intimate interface with underlying operating system necessitate the use of a low level language such as C). However, if we know a program such as `compare0` is going to be re-implemented later in a lower level language we would think twice. This is because we know that high level primitives like encode (`<.`) and iota (`!`) are not going to be there in C or an assembler. An explicit effort not to use such primitives is what we refer to as *low level programming*. A low level version of the comparison function is the following:

```
@.z<-lkey compare rkey
  for (x:!8)
    case (*(alph!lkey[x])-alph!rkey[x]) {
      _1: z<-1
          ->0
       1: z<-1
          ->0
```

```
    }
   z<-0
@.
```

We note that the function `compare` compares one pair of characters at a time (similar to compare one bit at a time in a ***radix*** sort where the keys are binary bit-strings), and yield a result as soon as the order can be determined where as in `compare0` a global calculation involving all characters in a key is always carried out. Clearly, the second comparison function does less work. But due to interpreter overhead we mentioned earlier the second function may execute slower than the first in most cases in APL/ELI, unless the code is compiled by an APL/ELI compiler. However, for prototyping purpose, the second one is what we want as it avoids the use of two high level primitives.

3.2.2 tree operations

Suppose a tree already exists. Given a key `key` we would like to insert it into the tree or find the tree node (node index) where the key have been placed. The algorithmic outline is

if *the tree is* ***nil then*** *place the key at root* ***else*** *compare with the root node:*
if *key equals to the one at root* ***then*** *found* ***else*** *search the left subtree or the right subtree*

and the ELI function is

```
@.z<-tree search key
  if(tree=_1) {z<-inct
    tkey[z;]<-key}
  else
    case (*tkey[tree;] compare key) {
      0: z<-tree
      1: z<-tkey[tlson[tree];] search key
     _1: z<-tkey[trson[tree];] search key
    }
@.
```

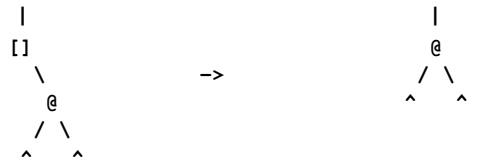
Let `nmat` be a character matrix. Then the following function will build a tree out of `nmat`:

```
@.z<-buildtree nmat
  z<-search nmat[0;]
  for(x: 1+!_1+1^.#nmat) search nmat[x;]
@.
```

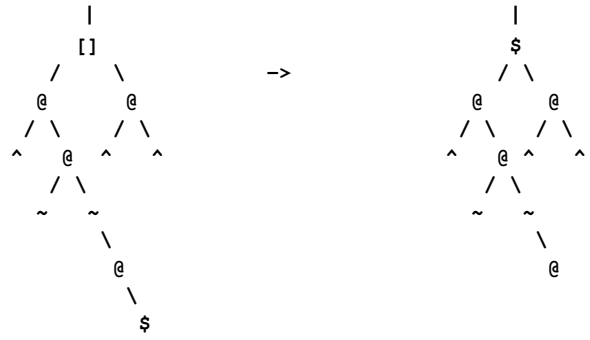
To delete an item with key `key` from a tree, we use a recursive scheme similar to that in search. But we need to do some cutting and reattachment when a node with the same key is found and removed. The basic algorithm is:

if *the tree is nil then item not found*
else *compare with the root node:*
  if *key equals to the root node then*
    if *left subtree if nil then return right subtree*
    else
      (find the right most descendent of left subtree,
       detach it and put it in the place of the root)
  else *search the left subtree or the right subtree*

Line 4 above is easy to understand (where `[]` represents the node to be deleted and `^` represents subtrees):



Lines 6 and 7 can be illustrated by the following diagram:

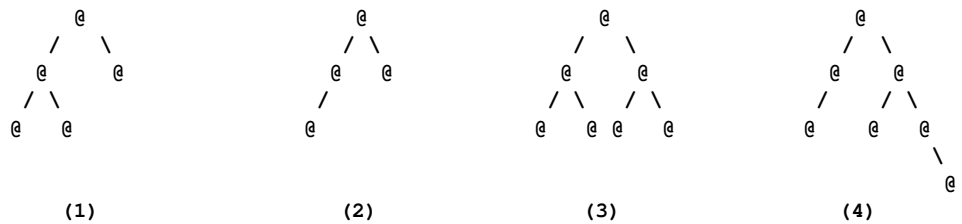


The reason for this reattachment is because the organizing principle of a binary tree is that at any node `n`, the key associated with it is **>** all keys associated with the nodes on the left subtree and is **<** all keys associated with the nodes on the right subtree. When the node **n** (**[]**) is deleted the right most node of the left subtree has this property. Therefore, it becomes a natural candidate to replace the deleted node. The program is a bit more complicated than the `search` program. For one thing we need to remember the parent of the node to be deleted and whether that node is a left or right son of that parent. Hence, the left argument `ptree` of the function is a (*parent*, *current-root*) couple while the first character of the right argument is either '`l`' (current node is a left son) or '`r`' with the rest a key; so the two lines of code using the execute function '`!.`' (see sect.1.6) is simply a shorthand for

either `tlson[..]<-..`
or `trson[..]<-..`

```
@.z<-ptree delete skey
  p<-ptree[0]
  tree<-ptree[1]
  side<-skey[0]
  key<-1!.skey
  if(tree=_1) ->'no node matches ',key
  else
    case (*tkey[tree;] compare key) {
      0: if (_1=ls<-tlson[tree]) !.'t',side,'son[p]<-tlson[tree]'
         else {
           lp<-ls
           while (_1~=rs<-trson[ls]) //go down right side of subtree
             {lp<-ls
              Ls<-rs}
           !.'t',side,'son[p]<-tlson[tree]'  //attach to parent of deleted node
           trson[ls]<-trson[tree]            //attach original right subtree
           if (lp~=ls) trson[lp]<-_1         //detach from subtree's right most branch
         }
       1: z<-tkey[tlson[tree];] delete key
      _1: z<-tkey[trson[tree];] delete key
    }
@.
```

The reason we want to organize data in a tree structure is for ease of searching, insertion and deletion. A tree node `n` is an *internal* node if it has a left son or right son, i.e. `tlson[n]~=_1` or `trson[n]~=_1`. A binary tree is ***full*** if all its internal nodes have two sons. A tree is ***perfectly balanced*** if at any internal node, the difference in the number of nodes in its two subtrees are less than or equal to 1. For example,



Tree (1) is full but not perfectly balanced. Tree (2) is perfectly balanced but not full. Tree (3) is full and perfectly balanced. Tree (4) is neither. The ***height*** of a tree is the maximum number of nodes from the root to a leaf node. In a full and perfectly balanced tree of **`n`** nodes, approximately half of the nodes are leaf nodes and half are internal nodes (see tree (3)). And the height of such a tree is **`log n`**. That means we only need to make **`log n`** comparisons to locate an item instead of **`n`**. However, a tree can be such that each internal nodes has only one child. Then a search in such a tree becomes linear such. For example, if `nmat` in the above function is the matrix `t` in section 2.1, we then have such a skinny tree. The reason that this will result in linear tree is because `t` happens to be lexicographically ordered to start with and the function `buildtree` builds the tree top-down. The following function builds a perfectly balanced tree recursively for `n` number of nodes from bottom up, i.e. nodes of subtrees are allocated before its root.

```
@.z<-buildtree2 n
  if (n=0) z<-_1
  else {
    nr<-_1^.n-nl<-~.n%2
    tkey[z<-inct]<-nmat[x<-x+1]
    tlson[z]<-buildtree2 nl
    trson[z]<-buildtree2 rl
  }
@.
```

And to build a tree from as a set of keys for tree nodes, we do

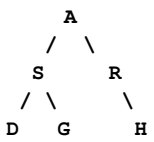
```
x<-nxtndx
buildtree2 n<-1^.#nmat
```

We note that the variable `x` in line 4 of the second tree building function is a global variable and need to receive a value before the function can be called. Finally, we observe that although the tree built by `buildtree2` is perfectly balanced, it does not have the ordering property of a tree built by `buildtree` that for any node `n`, its key is greater than any key (lexicographically) in its left subtree, and less than those on the right subtree. To build a tree which has the ordering property and more balanced than simply using insertion function `search` requires more sophistication and we will not get into details here. Instead a reader can see Wirth [14] section 4.4.7.

3.2.3 tree transversals

A *tree* ***traversal*** is a way to visit nodes in a tree in certain order. In contrast, a *search* only aims at finding some particular node in a tree. There are basically three ways to traverse a binary tree:

1. ***preoder****: root, left subtree, right subtree*
2. ***inorder****: left subtree, root, right subtree*
3. ***postorder****: left subtree, right subtree, root*



For example, a *preorder* traversal of the tree above is

```
A S D G R H
```

An *inorder* traversal of the tree above is

```
D S G A R H
```

and a *postorder* traversal of the tree above is

```
D G S H R A
```

The (recursive) functions to do traversal are as follows where parameter `tree` is an index to arrays representing trees:

```
@.preorder tree
  if (tree~=_1){
    []<-tkey[tree;]
    preorder tlson[tree]
    preorder trson[tree]
  }
@.

@.inorder tree
  if (tree~=_1){
    inorder tlson[tree]
    []<-tkey[tree;]
    inorder trson[tree]
  }
@.

@.postorder tree
  if (tree~=_1){
    postorder tlson[tree]
    postorder trson[tree]
    []<-tkey[tree;]
  }
@.
```

However, many times we would like to have a non-recursive version of an algorithm. For example, if we are prototyping for a tree traversal to be later implemented in an assembly language, or we would like to avoid many

function calls in consideration of execution performance. Let us illustrate how we can design a non-recursive version using the function `postorder` as an example. The basic algorithm to get a non-recursive postorder traversal is as follows.

go down to the left most node **n**
visit the right subtree **rst** *whose parent is* **n**
visit **n**
replace **n** *by its parent* **p** *and continue until root is reached*

We want to take this opportunity to discuss the interplay between data structures and algorithms. Although it is possible, as we will show subsequently, to construct an ELI/APL program to implement the above algorithm using tree data structure we introduced earlier, the program will be greatly simplied if our tree data structure has an additional field:

```
tpare
```

which is initialized to all `_1` and for each node `n`, `tpare[n]` is the parent node of `n`. Now assume this field has been incorporated into our tree implementation in the beginning, our first attempt in getting a non-recursive version in ELI which still use recursion partially is the following:

```
@.nrpostorder0 tree;n;n0;rs
  if (tree~=_1)->0
  n<-tree
  while (_1~=n0<-tlson[n]) n<-n0
 //n is the left most node at this point
up:if (_1~=rs<-trson[n]) nrpostorder0 rs  //process right subtree
   []<=tkey[n;]
   if (n~=tree) {n<-tpare[n]       //climb up tree if not the root
     ->up
   }
@.
```

While the introduction of the field `tpare` helps us in this program, we must point out that once this field has been incorporated in our tree data structure, we need to incorporate it both in the `search` and `delete` functions. And when we take a more careful look into how to make necessary changes in these two functions, we realize that it is not entirely trivial. Hence, we decide to give up the idea of introducing an additional field to indicate the parent of a node. Instead, we use a list `path` to remember the path when we go down from the root of a tree's left most node:

```
@.nrpostorder1 tree;n;path
  if (tree~=_1)->0
  n<-tree
  path<-!0
  while (_1~=n0<-tlson[n]) {path<-n,path
    n<-n0
  }
 //n is the left most node at this point
up:if (_1~=rs<-trson[n]) nrpostorder1 rs  //process right subtree
   []<=tkey[n;]
   if (n~=tree) {n<-path[0]         //climb up tree if not the root
     path<-1!.path
     ->up
   }
@.
```

We have reduced one recursive call from a total of two in the original algorithm. To eliminate the recursive call in traversing right subtrees when climbing up from left most node, we further introduce a stack of roots of subtrees,

rootstk, and check nodes against the top of this stack instead of the original root tree. A ***stack*** is a data structure with the discipline that the element which joins the stack last is the first to be popped off the stack. To implement a stack stk in ELI/APL we simply do

```
stk<-!0       // initialize it to an empty stack
stk <-n,stk   // put item n on top of a stack
n<-stk[0}     // get the top of a non-empty stack
stk<-1!.stk   // pop the top of a stack
```

In fact, the variable path we introduced above to remember the path we go down when descending a tree is a *stack*. In the following we further expand this to paths which remembers go-down paths in multiple (sub-)tree descends. The function is

```
 @.nrpostorder tree;n;paths;p;rs
   if (tree~=_1)->0
   rootstk<-,n<-p<-tree
   paths<-pstk<-!0
   while (_1~=n0<-tlson[n]) {path<-n,path
     n<-n0
   }
 //n is the left most node at this point
up:if (_1~=rs<-trson[n]) {        //prepare to process right subtree
      rootskt<-rs,rootstk
      pstk<-(n<-p),pstk
      while (_1~=n0<-tlson[n]) {path<-n,path
        n<-n0
        ->up
      }
   }
   []<=tkey[n;]
   if (n~=rootstk[0]) {n<-path[0]  //climb up tree if not the root
     path<-1!.path
     ->up
   } else if (0~=#rootstk<-1!.rootstk) {
     n<-pstk[0]
     pstk<-1!.pstk
     ->up
   }
 @.
```

While we put together a non-recursive version of postorder without introducing the field tpare, sometimes it is both useful and convenient to have that field build into the tree data structure. This is particularly true in cases where going from a child node to its parent node is frequent. For example, if we use trees to represent expressions in a program for analysis, i.e. parse trees, where subtrees correspond to sub-expressions and we need to find the parent of a node during code generation.

3.3 Quad Trees

Quad trees are used in digital image processing. A quad tree is of the following PASCAL record:

```
type qtree = record
               color: char;
               sonnw, sonne, sonsw, sonse: ↑qtree;
             end;
```

A variable of type qtree represents the image of an area in the following way:

```
iv.color= 'b':  the area is totally black
iv.color= 'w':  the area is totally white
iv.color= ' ':  the area is grey and sonnw, sonne, sonsw, sonse describe the image
                of its four quandrants recursively
```

Using our standard implementation of records, we have the following to initialize a quad tree:

```
@.initqtree
  qtcolor<-(incqtree<-maxqtx<-1000)#' '
  qtsons<-(maxqtx,4)#_1
  qtcolor[0]<-'w'
  qtcolor[1]<-'b'
  nxtqtx<-2
@.
```

Note that the first two node `0` and `1` are set to be all white and all black images. For any quad tree (index) `t`, `qtsons[0]`, `qtsons[1]`, `qtsons[2]`, `qtsons[3]` represent the quad tree indices of its four quadrants. To create a new quad tree, we need the following function:

```
@.incq
  if (maxqtx>=nxtqtx<-1+nxtqtx)->0
  else {  // increase the table by a chuck of size incqtree
    qtcolor<-qtcolor,incqtree#' '
    qtsons<-qtsons,.(incqtree,4)#_1
  }
@.
```

We would like to ***union*** two quad trees `s` and `t` such that `s union t` represents the image which is a coagulation of the two images represented by `s` and `t`. The algorithm for *union* is the following:

if `s` *or* `t` *is the black qtree then return the black qtree*
if `s` *is white then return* `t`
else if `t` *is white then return* `s`
else
(for i=0..3 do
`u[i]= qtsons[i] union qtsons[i]`
if all `u[i]` *are black then return black)*

We note that the last line in the algorithm description is aimed at situations that when all four quadrants of an image area, after union, become black, then the image itself becomes black. In the following function, `s` and `t` are two quad tree indices. And remember that `0` is the all white image and `1` the all black image.

```
@.s union t
  if ((s=1)&t=1) z<-1
  else if (s=0) z<-t
  else if (t=0) z<-s
  else {z<-incqt
    for (i:!4) qtsons[z;i]<-qtsons[s;i] union qtsons[t;i]
    if (^/1=qtsons[z;]) z<-1
  }
@.
```

Finally, we notice that the original field `tcolor` for indicating 'b', 'w' or ' ' is not really needed as long as we keep the convention that `0` is the all white and `1` is the all black quad tree.

3.4 Graph Algorithms

3.4.1 graph representations

Binary trees and quad trees are all specific types of graphs. In this section we consider algorithms on more general graphs. A ***graph*** *G* is a tuple (*V, E*) where *V* is a set of *vertices* v_0, v_1, … and *E* is a set of *edges* $\{(v_i,v_j)\}$. Each node (vertex) can carry additional information but we assume it is just a point. Here we exclude self-pointing edges and multiple edges from one graph node to another. We also assume a graph is directed, i.e. an edge (v_i,v_j) is from vertex v_i to vertex v_j, and can carry a *weight* w_{ij} (for example, it may indicate the distance from one city to another) which can be just all `1` for graphs with no weight consideration. We also assume that a graph is *connected*, i.e. from any pair of two vertices $\{v_i, v_k\}$ there is always a path from one vertex to the other.

Graphs are used extensively in modeling and analysis from travelling planning, network analysis and program dataflow analysis. The sizes and densities (the proportion of `e` the number of edges to `n` the number of vertices) vary extremely. A graph is called ***sparse*** if `e` is not proportional to n^2. For example, a graph representing a social network of millions/billions users of a service is sparse while a flow graph of a program is likely not sparse. There are basically two ways to implement a graph `G`. An *adjacent* ***matrix*** `m` of a graph is a `n` by `n` matrix where `m[i;j]` is `0` or it is the weight of an edge from v_i to v_j or `1`. An *adjacent* ***list*** is a triple `L=(I;V;W)` where `I` is a vector of indices, from `0` to `n-1`, of all vertices v_i in `G` with outward edges, `V` is a list with each i-th item the collection of indices j such that edge (v_i,v_j) exists in `G`, `W` is the list of weights w_{ij} corresponding to `V` or empty for an un-weighted graph. For graphs of moderate size such as program flow graphs, matrix representation is quite convenient. On the other end, for huge sparse graphs, adjacent list representation is necessary due to storage considerations.

Let us consider a graph `G` given as an `n` by `n` adjacent matrix which is not weighted (i.e. a boolean matrix), how can we transform it into an adjacent list representation `L=(I;V)`. We remind here that we assume through this chapter that `[]IO=0`. First, for a node index `i` to be in in the vector `I` it must have at least one outgoing edge v_{ij}, i.e. `1=&/G[i;]`. Hence, `I` is `(&/G)/!n`, and for each `i` in `I`, the corresponding item in `V` is `G[i;]/!n`. So, we have the following function:

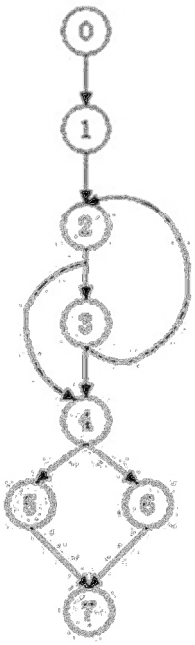
```
@.z←mat2list G;n;I;V;ix
  n←-^I←-(&/0~=G)/g←-!1^.#G
  V←n#_
  for (ix:0;n-1) V[ix]←G[I[ix];]/g
  z←-(I;V)
@.
```

We can also convert an adjacent list of a graph into an adjacent matrix representation. We'll leave that for our reader. We note here that *list* as an extension to *array* in classical APL is a useful way to deal with non- rectangular data such as the collection of target vertices `V` here, and the use of the ***for*** statement makes the code more readable.

To illustrate the above, we have the matrix and list representations of the graph `G` depicted in the diagram below.

```
      G←-8 8#[]IO←-0
      G[0;1]←-G[1;2]←-G[2;3 4]←-G[3;4 2]←-G[4;5 6]←-G[5 6;7]←-1
      G
0 1 0 0 0 0 0 0
0 0 1 0 0 0 0 0
0 0 0 1 1 0 0 0
0 0 1 0 1 0 0 0
0 0 0 0 0 1 1 0
0 0 0 0 0 0 0 1
0 0 0 0 0 0 0 1
0 0 0 0 0 0 0 0
```

```
      mat2list G
<0 1 2 3 4 5 6
<<1
 <2
 <3 4
 <2 4
 <5 6
 <7
 <7
```



3.4.2 depth first search

Given a graph `G`, a ***depth first search*** visits nodes of `G` from starting node `s` until all nodes reachable from `s` are visited in such a way that in each path it goes as far as it can before taking a new path. We follow the algorithm as expressed in Figure 6.9 of Heineman-Pollice-Selkow [8]. The basic idea is for the main function `DFS` to recursively call `dfs_visit` when visiting nodes of `G` reachable from the start node `s`. During the process, we assign a color to each node:

white	node has not been visited
grey	node has been visited, but it may have an adjacent node that has not been visited
black	node has been visited and so have all its adjacent nodes

All vertices are initially colored *white*, and upon visiting a vertex `v` we immediately color it *grey*. We maintain a global incrementing counter `counter` when visiting nodes of `G`. To keep track of the search, we introduce three variables:

`pred[v]`	the predecessor vertex that is used to recover a path from the source vertex `s` to vertex `v`.
`d[v]`	the value of `count` when depth-first search first visited `v`.
`f[v]`	the value of `count` when depth-first search finished visiting `v`.

The ELI code is as follows where we assume that the graph `G` is an adjacent (boolean) matrix form and vertices are indicated by integers such as the start node `s`.

```
@.G depthFirstSearch s;counter;d;f;n
  counter<-[]IO<-0
  d<-f<-pred<-(n<-1^.#G)#_1
  color<-n#`white
  dfs_visit s
  for (v:0;n-1)
    if (color[v]=`white) dfs_visit v
@.

@.dfs_visit u
  color[u]<-`grey
  d[u]<-counter<-counter+1
  for (v:G[u;]/!n) {
    if (color[v]=`white) {
      pred[v]<-u
      dfs_visit v
    }
  }
  color[u]<-`black
  f[u]<-counter<-counter+1
@.
```

We see that the code is very close to the pseudo code Figure 6-9 in [8]. We also refer to [8] for analysis of the efficiency of this algorithm.

3.4.3 single-source source shortest path

The well-known *shortest path problem* is that given a directed weighted graph G, find the shortest path from a starting vertex s to a destination vertex d. This problem has wide applications such as in a map G of available flights between cities with weight representing distance; one would like to find a path of shortest distance from city s to city d. For simplicity, we only consider a single-source vertex, and we assume that `[]IO=0` and s=0. We assume that the input to the problem is a weighted directed graph G in adjacent matrix form, and follow the presentation in Figure 6-13 in [8] of the Dijkstra's algorithm for solving the shortest path problem. The algorithm produces two vectors dist and pred over vertices of G; for a vertex v, dist[v] is the *shortest distance* from s(=0) to v (hence dist[0]=0) and pred[v] is the *predecessor* of v in a path for shortest distance.

The essence of the algorithm is to expand a set S of vertices in greedy fashion, for which the shortest path from 0 to every vertex v in S is known but only using paths that include vertices in S. Initially, S={0}. To expand S, the algorithm finds the vertex in V-S (i.e. a vertex of G not in S) whose distance from s is the smallest, and follows v's edges to see whether a shorter path exists to another vertex. We refer to graph in Figure 6-14 in [8] for illustration, and we see after processing v_2, the algorithm determines that the distance from s to v_3 is really 17. Once S expands to V, the algorithm complete. The pseudo-code for the algorithm (Figure 6-13 in [8]) is as follows:

```
singleSourceShortestPath (G,s)
    PQ= new Priority Queue
    foreach v in V do {dist[v]=inf; pred[v]=-1}
    dist[s]=0
    foreach v in V do insert (v,dist[v]) into PQ
    while (PQ is not empty) do
        u=getMin(PQ)
```

```
        foreach neighbor v of u do
            w=weight of edge (u,v)
            newLen= dist[u]+w
            if (newLen<-dist[v]) then
               decreaseKey (PQ, v, newLen)
               dist[v]= newLen
               pred[v]=u
    end
```

A *priority queue* PQ is a queue Q for which when a new arrival joins Q it is inserted into Q according to some priority measure, is our case the `dist[v]`, i.e. the shorter the distance more ahead it will be placed in the queue; *pop* of PQ is the same as ordinary queue by just removing the head item from the queue. In the C++ version in [8], two library packages (`Graph`, `BinaryHeap`) and relevant function calls are deployed to implement the above. In ELI, we have a much simpler code: PQ is set to be an array initially has the source `0` and *priority* is maintained by keeping `dist[PQ]` to be a non-decreasing vector, i.e. ones with smaller `dist` than `newLen` are placed ahead of `v`. For the ELI code below, we just note that `PQ[0]` is the first element of `PQ` and `PQ<-1!.PQ` is the *pop* of `PQ`. The for-statement is ranging over vertices which are targets of `u` other than `0`. The extra if-statement in code is for the case that `v` may be in `PQ`; `inf` is represented by a largest possible distance.

```
      @.ssShortstp G;n;V;inf;PQ;du;v;u;newLen;b;PQ0
      //0 is the single source. G is a weighted directed adjacent matrix. []IO=0
        V<-!n<-1^.#G              //n is the number of nodes in G
        dist<-n#inf<-1+n*~./~./G
        dist[0]<-0
        pred<-n#_1
        PQ<-,0
        while (0!=#PQ) {
          du<-dist[u<-PQ[0]]
          PQ<-1!.PQ
          for (v:(0~=G[u;])/V) {
            newLen<-du+G[u;v]
            if (newLen<dist[v]) {
               if (v?PQ) PQ0<-(~PQ=v)/PQ
               else PQ0<-PQ
              //insert v into PQ so the order of magnitude of dist[PQ] is maintained
               PQ<-((~b)/PQ0),v,(b<-newLen<=dist[PQ0])/PQ0
               dist[v]<-newLen
               pred[v]<-u
            }
          }
        }
      @.
```

For the input sample given in Figure 6-14 of [8], we have

```
      []IO<-0
      G<-6 6#0
      G[0;1 3 2]<-6 18 8
      G[1;4]<-11
      G[2;3]<-9
      G[4;5]<-3
      G[5;3 2]<-4 7
ssShortstp G
      !6
0 1 2 3 4 5
      dist
0 6 8 17 17 20
```

```
      pred
_1 0 0 2 1 4
```

We note that in contrast to C++, ELI's code is much self-contained, succinct and close to the pseudo-code description; it is easy to understand as natural operations on arrays. We also notice that the for-statement in ELI is more powerful than that in C, and the convenient way to insert v into the priority queue PQ is achieved by using ELI primitives provided for splitting arrays according to a Boolean predicate and glue pieces together in a dataflow style.

4. Computational Algorithms

4.1 Iterative Method

We consider a simple heat dissipation problem (APL version of the solution is in example 3 of [6]). Giving a set of temperatures at the boundary *B* of a rectangle *R*, compute the eventual temperature at *R* based on the fact that the temperature at a point is the average of its four neighbors. Suppose *R* is represented by a grid of m by n points, then a typical C-style solution is to iterate over a double loop of *i* and *j* of the kernel (resulting in a triple loop)

```
newt[i;j]= (t[i-1;j]+t[i+1;j]+t[i;j-1]+t[i;j+1])*0.25       i=1,..,m; j=1,..,n
```

until the difference between `t` and `newt` is less than some preset threshold *e* at all point in *R*. Of course we can do similarly in ELI. But the array-oriented nature of ELI suggests another interesting approach. Instead of computing new temperate at each point by averaging the temperature of its neighbors we rotate array *A*, which denotes the temperature over *R*, one row up, one row down, one column to the left, one column to the right and compute average the four rotated planes to get a set of new temperatures (see the rotating example in sect.1.7). We can see that this new method has the same effect as the C-style solution listed above. The difference is that now we have a single loop instead of the initial triple loop.

We put all these into a function called *jacobi*. The left argument `f` is a Boolean matrix with `0`s on the boundary and `1`s at the interior, the right argument `a` is a floating point matrix of the same shape as that of `f`, denoting initial temperatures with interior points all `0`; is the final result temperature which depends on the threshold `e` set in the first line. We note that `~f` is the negation of `f`, i.e. with all `1`s at boundary and all `0`s at the interior.

```
     @.z<-f jacobi a;c;e;cnt
       e<-0.1+cnt<-0
       c<-(z<-a)*~f    //z,c get a as initial value
     L:r1<-_1$.a<-z    //up rotate by 1 row
       r2<-1$.a        //down rotate by 1 row
       r3<-1$a         //right rotate by 1 column
       r4<-_1$a        //left rotate by 1 column
       cnt<-cnt+1
       ->(e<~./|,a-z<-c+0.25*f*r1*r2*r3*r4)/L //taking average and diff
       cnt
      @.
```

In the line before the last, after taking the difference between the new value and old value `a-z`, we ravel `,a-z` it into a vector to get the absolute value `|` of the difference, and then takes the maximum `~./` of that vector to check to see if it is still larger than `e`. If it is, go back to loop head `L`, otherwise print out the value of `cnt` and exit. We note that the variable `cnt` is not essential, but lets us know how many iterations we have go through to get within the required limit for convergence. We make some small sample run.

```
      f<-5 7#1
      a<-5 7#0
      f[1 5;]<-f[;1 7]<-0
      a[;1 7]<-5 2#10 2.5 9.1 11.3 7 3.8 4.5 20.1 19.2 5.4
      a[1 5;1+!5]<-2 5#11.1 23.3 5.8 7.2 0.3 4.9 14.9 9.2 16 2.7
      f
0 0 0 0 0 0 0
0 1 1 1 1 1 0
0 1 1 1 1 1 0
0 1 1 1 1 1 0
0 0 0 0 0 0 0
      a
```

```
10   11.1 23.3 5.8  7.2 0.3  2.5
 9.1  0    0   0    0   0   11.3
 7    0    0   0    0   0    3.8
 4.5  0    0   0    0   0   20.1
19.2  4.9 14.9 9.2 16   2.7  5.4
      f jacobi r<-a
16
10   11.1         23.3         5.8          7.2          0.3          2.5
 9.1 10.56506618  13.35146139  9.162343675  7.997351304  6.743705601 11.3
 7    8.792694442 10.56761059  9.668714823  9.071017288  7.461734274  3.8
 4.5  7.193014314 10.66325575 10.18157327  11.36247899  10.38498706  20.1
19.2  4.9         14.9         9.2         16            2.7          5.4
```

We see that it takes 16 iterations to get interior temperature to converge within `0.1`. We would like to make `e` to be a parameter so we can see how a change of that limit effect the number of iterations required. Thus the limit `e` becomes the left parameter and the original left and right parameters are combined to be a list `(f;a)` as the new right parameter in our newer version. We take this opportunity to eliminate the intermediate variables r_i; and we modify the last line slightly to print a header for the count variable `cnt` by glue it with format function `(+.)` applied to a number to get its character representation.

```
[0]   z<-e jacobi1 (a;f);c;e;cnt
[1]   cnt<-0
[2]   c<-(z<-a)*~f
[3]   L:cnt<-cnt+1
[4]   ->(e<~./|,a-z<-c+0.25*f*(_1$.a)+(1$.a)+(1$a)+_1$a<-z)/L
[5]   'count: ',+.cnt
      0.1 jacobi1 (f;r)
count: 16
10   11.1         23.3         5.8          7.2          0.3          2.5
 9.1 10.56506618  13.35146139  9.162343675  7.997351304  6.743705601 11.3
 7    8.792694442 10.56761059  9.668714823  9.071017288  7.461734274  3.8
 4.5  7.193014314 10.66325575 10.18157327  11.36247899  10.38498706  20.1
19.2  4.9         14.9         9.2         16            2.7          5.4
       0.01 jacobi1 (f;r)
count: 26
10   11.1         23.3         5.8          7.2          0.3          2.5
 9.1 10.66631246  13.53233886  9.364624216  8.177725139  6.844740327 11.3
 7    8.94053114  10.81550314  9.963675573  9.318610663  7.608858719  3.8
 4.5  7.294261178 10.84413374 10.38385498  11.54285334  10.48602238  20.1
19.2  4.9         14.9         9.2         16            2.7          5.4
```

We can even put this function in a more elegant form using control structures of ELI to avoid old APL fashioned branching:

```
@.z<-e jacobi2 (f;a);c;cnt
  cnt<-1
  c<-(z<-a)*~f    //z,c get a as initial value
  while (e<~./|,a-z<-c+0.25*f*(_1$.a)+(1$.a)+(1$a)+_1$a<-z) cnt<-cnt+1
  'count: ',+.cnt
@.
```

It is clear that array-oriented programming results in a very different style of program than traditionally C-style loop-based program. Historically APL community developed array-oriented programming not just because APL language encourages such thinking but also because the execution inefficiency of the APL interpreter severely punishes loopy and non-succinct code. While the arrival of a good compiler can lessen such penalty it is precisely programs of such style being most suitable for effective automatic parallelization by a parallelizing compiler without introduction of additional parallel constructs alien to natural mathematical thinking in problem solving (see [6]).

4.2 Simple Encryption and Monte Carlo Method

4.2.1 encryption by random permutation

Recall the *dyadic* function ***deal*** `A?.B`, both `A` and `B` positive integers with `A<=B`; `A?.B` picks `A` *distinct* numbers from `!B` we introduced in section 1.5 (please **remember** there is a '**.**' after '`?`' for random number generators)

```
      10?.100
41 68 58 93 84 52 9 65 41 70
```

Now suppose `A=B` in above, say `26`; the deal function would then result in a permutation of `!B`.

```
      perm<-26?.26
      alph<-'abcdefghijklmnopqrstuvwxyz'
      alph[perm]
trqbpfisdnoemwvlgjkhcyzaxu
      scrb<-alph[perm]
```

So we use a random permutation generated by an application of the deal function to scramble a text string as a simple encryption tool. But how do we decrypt an encrypted text string at the receiving end? To this end we introduce the primitive monadic sort function ***grade up*** (`<`) in ELI: for vector `V`, `<V` is a *permutation* of the indices of `V` so that `V[<V]` is in a *non-decreasing* order.

```
      V<-52 84 4 6 53 68 1 39 7 42
      <V
7 3 4 9 8 10 1 5 6 2
      V[<V]
1 4 6 7 39 42 52 53 68 84
```

The grade up `<perm` of the encryption permutation is the decryption key:

```
      scrb[<perm]
abcdefghijklmnopqrstuvwxyz
```

Now consider a more general case of a very long text string: to send a very long decryption key which is of the length of the text is certainly not practical. A reasonable solution is to designate a set of distinct characters as *core* to be scrambled and leave characters in the string which do not belong to the core to be left unchanged. To this end first let us work out the case that the plain text is a string of characters all belong to the core but with possible repetitions and not every character in core is in it. Say core is the lower case alphabets plus the space.

```
      #core<-' abcdefghijklmnopqrstuvwxyz'
27
      perm<-27?.27
      perm
4 21 13 15 6 2 19 26 11 23 1 12 16 3 18 25 8 20 9 27 10 7 24 5 14 17 22
      ptxt<-'i like to see you'
      etxt<-core[perm[core!ptxt]]
      etxt
vcovkacixczaacpxf
```

where `core!ptxt` are positions of characters in `ptxt` in `core` and `etxt` is the encryption of `ptxt`. To decode it, we just apply the grade up function `<` to `perm` as before with encrypted core `ecore` replacing `core`:

```
      ecore<-core[perm]
      ecore
ctlnearyjv kobqxgshzifwdmpu
```

```
      ecore[(<perm)[ecore!etxt]]
i like to see you
```

Return to our original case that we have a large text `lptxt` with characters out of `core`, we need to remember the boolean vector `w` indicating which elements of `lptxt` are in `core` first to reduce `lptxt` to those in `core` for encryption and later to find positions in resulting string to be replaced by scrambled text. We illustrate this process with a sample `lptxt`:

```
      lptxt<-'I like to see you at 11:00 pm today!'
      iptxt<-(w<-lptxt?core)/lptxt   //get those chars of lptxt which are in core
      w
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 1 1 1 1 0
      iptxt
 like to see you at  pm today
      core[perm[core!iptxt]]         //scramble iptxt
covkacixczaacpxfcticcgbcixetp
      letxt<-lptxt
      letxt[?w]<-core[perm[core!iptxt]]  //?w indicates where iptxt sits inside letxt
      letxt
Icovkacixczaacpxfctic11:00cgbcixetp!
```

`?w` indicates where elements of `iptxt` sit inside `letxt` and we replace them by scrambled text while leave the rest undisturbed. To decrypt, we first note that `w` also indicates which elements of `letxt` are in `core` and then we can just unscramble that portion of `letxt` to its original form:

```
      letxt?core
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 1 1 1 1 0
      w/letxt
covkacixczaacpxfcticcgbcixetp
      ietxt<-w/letxt
      ietxt
covkacixczaacpxfcticcgbcixetp
      ietxt[(<perm)[ecore!ietxt]]
avcaaaocataaaxceacoaaxiaocccx
      ecore
ctlnearyjv kobqxgshzifwdmpu
      ecore!ietxt
1 13 10 12 6 1 21 16 1 20 6 6 1 26 16 22 1 2 21 1 1 17 14 1 21 16 5 2 26
      (<perm)[ecore!ietxt]
11 3 21 12 5 11 2 13 11 18 5 5 11 8 13 27 11 6 2 11 11 26 25 11 2 13 24 6 8
      ecore[(<perm)[ecore!ietxt]]
 like to see you at  pm today
      letxt[?w]<-ecore[(<perm)[ecore!ietxt]]
      letxt
I like to see you at 11:00 pm today!
```

We encapsulate the above computation into two functions: *encrypt* and *decrypt*. For the first function, its right argument is a pair of variables `(core;perm)`, the core text to be scrambled and the random permutation used to encrypt, and its left argument is the text to be encrypted; the result is the encrypted text. For the second function, the right argument is the same as in the first function while the left argument is the encrypted message and the result is the recovered plain text. The first two lines of the function do suitability checking on the right argument.

```
@.z<-ptxt encrypt (core;perm);pl;w;iptxt
  if (1~=##core) ->'core must be a vector'
  if (~(pl=#=perm)^(#core)=pl<-#perm) ->'perm must of the same lgth as core with no duplicates'
  iptxt<-(w<-ptxt?core)/z<-ptxt
  z[?w]<-core[perm[core!iptxt]]
@.
```

```
@.z<-etxt decrypt (core;perm);pl;w;ietxt;ecore
  if (1~=##core) ->'core must be a vector'
  if (~(pl=#=perm)^(#core)=pl<-#perm) ->'perm must of the same lgth as core with no duplicates'
  ecore<-core[perm]
  ietxt<-(w<-etxt?core)/z<-etxt
  z[?w]<-ecore[(<perm)[ecore!ietxt]]
@.
```

Let us try with slightly different variables:

```
      #core<-' abcdefghijklmnopqrstuvwxyz0123456789'
37
      perm<-37?.37
      ptxt<-'Ms Evans, I like to see you at 11:00 pm today!'
       emsg<-ptxt encrypt (core;perm)
      emsg
MfaEysjf,aIadeztak aftta9 xaska22:qqaocak 0s9!
      emsg decrypt (core;perm)
Ms Evans, I like to see you at 11:00 pm today!
```

We note that repeat execution of `n?.n` in a workspace generates different permutations. So the encryption scheme can be looked upon as a one-time use pad. Still the need of send the permutation key to the receiving party of an encrypted message is not very safe. Hence modern encryption uses the RSA method based on public key encryption scheme. This method depends on the fact that knowing the product `n` of two very large prime numbers `p` and `q` it is computationally difficult to factorize `n` unless you know one of the primes.

4.2.2 Monte Carlo method to compute *pi*

Monte Carlo simulation is a computational method invented by Stanislow Ulam and John von Neumann during Manhattan Project. It is widely used in many situations where an analytic solution is not feasible such as in solving some partial differential equations and solving stochastic differential equations related to options pricing. At the heart of this method is using repeated random sampling to obtain numeric data. We illustrate this process by using the Monte Carlo method to find π, i.e. the value of `@1`.

In a 1 by 1 square Q, a quarter of the circle C inside Q consists of all points (x,y) such that $\sqrt{(x^2+y^2)}\leq 1$. The basic idea of the Monte Carlo method is to throw a large number of darts, say `m`, into Q and count the number of darts `n` which are inside C. Then the ratio `n/m` is approximately equal to the area of C, i.e. `(n/m)`=(1/4) π because the area of a circle is πr^2 where *r* is the length of the radius. Now the question is how do we generate a large number of *random* darts in the unit square? We recall that monadic scalar function ***roll*** `?.n` randomly generates a number between `[]IO` and `n-1+[]IO` in sect. 1.5. Based on the roll function, we have the function `rand n` in *standard.esf* which gives `n` independent random numbers inside the unit interval `[0,1]`:

```
     {rand: (?.x#1000000)%1000000}  //independently gen. x numbers from 1..1M, then divide by 1M
      rand 100
0.13153 0.75560 0.45865 0.53276 0.2189 0.04704 0.678865 0.679297 0.934693 0.383503 0.519417 0.830966
      0.03457 0.05346 0.52970 0.6711 0.00769 0.38341 0.066843 0.417486 0.686773 0.588977 0.930437
      0.84616 0.52693 0.09196 0.65392 0.416 0.701191 0.910321 0.762199 0.262453 0.047465 0.736082
      0.32823 0.63264 0.75641 0.99104 0.36534 0.24704 0.98255 0.722661 0.753356 0.651519 0.072686
      0.63163 0.88471 0.27271 0.43641 0.76649 0.47773 0.23777 0.274907 0.359265 0.166508 0.486518
      0.89765 0.90921 0.06056 0.90465 0.50452 0.51629 0.31903 0.986643 0.493977 0.266145 0.090733
      0.94776 0.07375 0.50071 0.38414 0.27708 0.91382 0.529748 0.464446 0.94098 0.050084 0.761515
      0.77021 0.82782 0.12537 0.01587 0.68846 0.86825 0.629544 0.736225 0.72541 0.999458 0.888573
      0.23319 0.30632 0.35102 0.51327 0.59111 0.84598 0.412081 0.841511 0.26932 0.415395 0.537304
```

To get random points in the unit square, we first generate their (x,y) coordinates, then compute their distance from the origin (0,0) by introducing a simple `length` function and count how many of them are inside the circle.

```
      x<-rand 100
```

```
      y<-rand 100
      {length:((x*.2)+y*.2)*.5}
length
      3 length 4
5
      3 4 length 4 5
5 6.403124237
      dist<-x length y
      #dist
100
      dist
0.843825767 1.02808002 0.970931958 0.865037832 0.6406978738 0.965754676 0.5154958158 0.5443669671
      0.647999475 1.11593778 1.15252348 0.707953201 1.15149743 1.07054072 1.10506844 0.8165047052
      0.7378729788 1.030025732 0.1318180786 0.8906919036 0.5924622512 0.4633327123 0.4193921507
      0.713929093 0.861922657 1.15128156 0.72301417 0.269342831 0.97007708 1.23955896 1.103319881
      0.5643415876 0.6954200043 0.5930147851 1.062624391 0.5442390188 0.4697911129 0.6666902581
      0.989885694 1.06079201 1.05174394 0.726557401 0.820142206 1.04891934 0.46025074 1.009132148
      0.58541043 0.67903196 0.78647025 0.464031383 0.9610964 0.783555152 0.7542674995 0.451436771
      0.755297341 0.6941582 0.941964123 0.654646899 1.1595364 0.750995766 0.95589978 0.4006316046
      0.219070924 0.764531777 1.09971674 0.47710968 0.25646163 0.95094821 0.56528018 0.9611551484
      1.297003305 0.4006022811 0.3143011801 0.6958074566 0.4627631845 0.8170429201 0.7954248035
      0.8898329278 0.5356466962 0.3997390532 0.4524425589 0.8313774474 1.354537315 0.6958495927
      0.847516659 0.69315699 0.99229927 0.650981749 0.878844029 0.953027188 0.52997251 1.15606853
      0.857055143 0.7911189 0.473287146 0.97634126 0.89622853 0.823066743 1.14533233 0.9795147438
      dist<=1
1 0 1 1 1 1 1 1 1 0 0 1 0 0 0 1 1 0 1 1 1 1 1 1 1 0 1 1 1 0 0 1 1 1 0 1 1 1 1 0 0 1 1 0 1 0 1 1 1
1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 0 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1
0 1
      +/dist<=1
79
      79%100
0.79
      (@1)%4
0.7853981634
```

To get more accurate result, increase the number of darts, and this is a general approach in Monte Carlo simulation:

```
      xl<-rand 10000
      yl<-rand 10000
      distl<-xl length yl
      (+/distl<=1)%#distl
0.783
```

4.3 Sparse Matrix Computation

We have stated in section 1.8 that the inner product `A+:*B` for two matrices `A` and `B` is just the common *dot-product* (matrix multiplication) of matrix `A` by matrix `B`, and for a vector `x`, `A+:*x` is the common *matrix-by-vector* dot-product where the width of `A` must equal to the height of `B` or the length of `x`. For simplicity, we assume the matrices we discuss in this section are all square matrices of size *n*. For example,

```
      A
1.2 0   0  3   4.8
0   2.1 5  0   0
0   0   1  3.8 7
0   2   0 12   0
0.8 0   0  0   1.3
      B
0 1 0 0 0
1 0 0 0 0
```

```
0 0 0 1 0
0 0 0 0 1
0 0 1 0 0
      x<-9.1 3 4.5 0.7 8
      A+:*B
0   1.2 4.8 0  3
2.1 0   0   5  0
0   0   7   1  3.8
2   0   0   0 12
0   0.8 1.3 0  0
      A+:*x
51.42 28.8 63.16 14.4 17.68
```

We notice that `B` is a column permutation of the identity matrix `I` (i.e. `I+:*A=A+:*I =A` for all square matrices `A`)

```
1 0 0 0 0
0 1 0 0 0
0 0 1 0 0
0 0 0 1 0
0 0 0 0 1
```

and consequently `A+:*B` is a column permutation of `A`.

In general, for size n, to carry out the dot-product `A+:*x` the ELI inner product `+:*` needs to perform n^2 multiplications and $(n\text{-}1)*n$ additions. Usually in ELI/APL programming a programmer is not excessively concerned about machine code execution efficiency while concentrating on the overall solution of a problem at hand. But in the case of operations on very large matrices this a crucial question to consider. First, for n larger than million or more, there is the question of how best to store the matrix in memory. In fact, many large matrices in real applications arising from solutions to partial differential equations (see Saad [12]) are ***sparse***, i.e. a large portion of matrix `A` of size n are `0` in the sense that the number of non-zero elements in `A` is proportional to cn, for some constant c, not n^2. This implies that we only need to store non-zero elements of `A` and we can also skip operations involve zero elements.

One of the methods to store nonzero elements of a sparse matrix is the *compressed sparse row* format (sect.3.4 [12]) or ***csr*** format. For a sparse matrix `A` of size n, this format comes with three vectors:

- a real (or complex) vector `v` of values of nonzero elements a_{ij} in `A`, row by row, with length cn.
- an integer vector `c` corresponding to vector `v` above with each element of `c` indicates the column number of corresponding element a_{ij}; `c` is also of length cn.
- an integer vector `p` of length n+1 where each `p[i]` points to the beginning in `v` of nonzero elements of row `i` for `i` from 1 to n and `p[`n`+1]=`cn+1.

For the specific matrix `A` in the example above, we have

```
      v<-1.2 3 4.8 2.1 5 1 3.8 7 2 12 0.8 1.3
      #v
12
      c<-1 4 5 2 3 3 4 5 2 4 1 5
      p<-1 4 6 9 11 13
```

To carry out a matrix-by-vector dot-product `A+:*x` of a sparse matrix `A` stored in *csr* format, we note that the i-th row `A` of nonzero elements of `A` is `v[p[i]..p[i+1]-1]` where for the range `p[i]..p[i+1]-1` we introduce a short function `range` in ELI so `x range y` will yield `x..y` as its result:

```
      {range:y+(-[]IO)+!1+x-y}
range
       2 range1 5
2 3 4 5
```

The corresponding row elements of `x` involved in the operation is then `x[c[p[i]range p[i+1]-1]]`. Hence, for a sparse matrix `A` stored in *csr* format (`v;c;p`) the following function carries out the matrix-by-vector operation for vector `x`:

```
[0]  y<-x spmatr_vec (v;c;p);n;idx;i
[1]  y<-(n<-_1+#p)#0  //initialize y
[2]  for (i:1;n){
[3]    idx<-p[i] range p[i+1]-1
[4]    y[i]<-v[idx]+:*x[c[idx]]
[5]  }

         x spmatr_vec (v;c;p)
51.42 28.8 63.16 14.4 17.68
```

A *compressed sparse column* format for a sparse matrix can be similarly defined but the function implementing the matrix-by-vector operation differs from the above in line 4 (see Saad [12] sect. 3.4).

Many sparse matrices have mostly nonzero elements on their diagonal, or diagonal elements need to be accessed more easily. Hence, for such a sparse matrix `A` of size *n*, we have a *modified sparse raw* (***msr***) format which has two vectors `dv` and `pc` of the same length. The first *n*-elements of `dv` stores the value of diagonal elements of `A`, `dv[n+1]` is not used, starting from position *n*+2, `dv` stores the nonzero elements of `A` excluding the diagonal, row by row. The first *n* elements of `pc` store pointers to `dv`, i.e. each `pc[i]` points to the starting position of nonzero elements of `A` in row `i` excluding the diagonal element with `pc[n+1]`.=1+*length* of `dv`. Starting from position *n*+2, each `pc[i]` stores the column index of the corresponding nonzero element `dv[i]` in certain row. We remark that in the rare case that if for certain row *i*, all elements are zero other than the diagonal element, we need to stick in a *0* in `dv` for that row with an arbitrary column number, say *1*, in the corresponding position in `pc`. This is to maintain the uniformity of processing code. For our given `A` above, we have the following vectors for its ***msr*** format:

```
      dv<-1.2 2.1 1 12 1.3 _999 3 4.8 5 3.8 7 2 0.8
      pc<-7 9 10 12 13 14 4 5 2 4 5 2 1
```

And the processing code for matrix-by-vector dot-product for a sparse matrix stored in *mrs* format is the following:

```
@.y<-x spmat_vec2 (dv;pc;n);idx;i
  y<-x*n^.dv
  for (i:1;n){
    idx<-pc[i] range pc[i+1]-1
    y[i]<-y[i]+dv[idx]+:*x[pc[idx]]
  }
@.
      X<-9.1 3 4.5 0.7 8
      X spmat_vec2 (dv;pc;n)
51.42 21.3 63.16 14.4 17.68
```

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